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AN APPLICATION OF KINETIC THEORY TO THE PROBLEMS OF EVAPORATION AND SUBLIMATION OF MONATOMIC GASES

By Prescott D. Crout

1. Introduction. In the simple case of a monatomic gas in equilibrium with its liquid in a closed vessel, the pressure exerted upon the surface of the liquid and upon that of the container is p_0, the saturation vapor pressure of the liquid at the temperature considered. From the standpoint of the kinetic theory, molecules are continually being emitted from the surface of the liquid into the gas, and are continually returning from the gas to the liquid; furthermore since the gas is homogeneous and isotropic, it follows that the speed distribution of the emitted molecules is the same as that of the incident molecules returning from the gas—the velocity distributions differ only in direction. It is thus evident that in unit time the momentum of the emitted molecules differs only in sign from that of the incident molecules, from which we see at once that the total pressure p_0 is made up of two *equal* parts: the recoil pressure of the emitted molecules, and the impact pressure of the incident molecules.

Let us next suppose that the liquid surface is held at the same temperature as before, but that each molecule is removed as soon as emitted. In this case we see that there are no molecules to return; hence the pressure acting on the liquid surface is merely the recoil pressure of the emitted molecules. If, now, we suppose that the velocity distribution of the emitted molecules is not affected by that of the incident molecules, which is equivalent to supposing the mechanism of evaporation to be such that secondary emission is a negligible factor, it follows that this recoil pressure is the same as before, namely $\frac{p_0}{2}$. We thus see that if the molecules are removed as soon as they are emitted, the surface pressure[1] is $\frac{p_0}{2}$. This case can be approached physically by having a cold condensing surface very close to the emitting surface; however we shall see later that it can *not* be approached by having a liquid evaporate in an evacuated vessel.

[1] Slepian, "Conduction of Electricity in Gases", p. 19.

We have now considered the extreme cases of zero and maximum rate of evaporation, and have seen that the surface pressure drops from p_0 to $\frac{p_0}{2}$ over this range; however these are but two points of the complete curve of surface pressure against rate of evaporation. The problem of obtaining this complete relationship is one of the problems treated in this paper.

It may be noted that in the case where the rate of evaporation is not zero, there is no reason to suppose that the gas is isotropic in the immediate vicinity of the liquid surface; hence in this vicinity the words "pressure" and "temperature" lose their ordinary significance. Further out from the liquid when the molecules have had sufficient time to collide with each other and again effect an equipartition of energy so that the gas is again isotropic, we may again speak of the "pressure" and of the "temperature" of the gas; however, this pressure is neither the saturation vapor pressure of the liquid, nor is it the liquid surface pressure. Obviously it is a function of the liquid temperature (or saturation vapor pressure) and the rate of evaporation, as are likewise the gas temperature and density. In the sections which follow, these functions are derived and tabulated; also it is shown that the rate of evaporation cannot exceed that rate which corresponds to a drift velocity equal to the velocity of sound in the gas.

If, now, we consider a system comprising an evaporating liquid discharging through a system of tubes and orifices into a known pressure, it is clear that such a system can be solved by means of the above mentioned functions. Examples of the solution of such problems are given in Section 10.

To illustrate the difficulties encountered by neglecting the deviation of the pressure and temperature of a gas just evaporated, from the saturation values for the liquid let us consider the one dimensional problem involved in the case of mercury evaporating from a hot liquid surface, and condensing on a parallel cooler surface. Such a condition could be closely realised in the case of a cylindrical tube if the ratio of diameter to length were sufficient to render the influence of the side walls negligible. Placing an x axis perpendicular to the mercury surface, the origin being at the hot surface, and placing

u = gas velocity parallel to x axis,

ρ = gas density,

p = gas pressure,

we see that u, ρ, and p are functions of x. But since force is the rate of change of momentum, we have the mechanical condition

$$\frac{dp}{dx} = -\rho u \frac{du}{dx}; \tag{1}$$

also for the conservation of matter we have the continuity equation

$$\frac{d}{dx}(\rho u) = \rho \frac{du}{dx} + u \frac{d\rho}{dx} = 0; \tag{2}$$

and finally we have the adiabatic equation of state of the gas

$$p = f(\rho), \tag{3}$$

which differentiated gives

$$\frac{dp}{dx} = f'(\rho) \frac{d\rho}{dx}. \tag{4}$$

We see that the functions u, ρ, and p must satisfy (1), (2), and (4) identically. Eliminating $\frac{dp}{dx}$ from (1) and (4), we have

$$f'(\rho) \frac{d\rho}{dx} = -\rho u \frac{du}{dx},$$

which divided by the expression

$$u \frac{d\rho}{dx} = -\rho \frac{du}{dx},$$

which is (2), gives

$$u^2 = f'(\rho).$$

Differentiating, we have

$$u \frac{du}{dx} = \frac{1}{2} f''(\rho) \frac{d\rho}{dx},$$

which substituted in (1) gives

$$\frac{dp}{dx} = -\frac{\rho}{2} f''(\rho) \frac{d\rho}{dx};$$

whence dividing by (4) we have

$$\frac{f''(\rho)}{f'(\rho)} = -\frac{2}{\rho}. \tag{5}$$

But *this is a condition upon the given equation of state of the gas*; hence it is impossible to satisfy it by suitably determining u, ρ, and p as functions of x. Integrating (5) we have

$$\log f'(\rho) = -2 \log \rho + c_1$$

$$f'(\rho) = c_2 \rho^{-2},$$

where c_1 and c_2 are constants. Integrating again, it follows that

$$p = f(\rho) = \frac{c_2}{\rho} + c_4 \tag{6}$$

is the form of the adiabatic equation of state of the gas required by (5); hence we see that (5) is not satisfied. In fact, in the case of a perfect gas the given equation of state is

$$p = f(\rho) = c\rho^{\gamma},$$

where c is a constant and γ is the ratio of the specific heats. Since γ is positive, this equation is evidently not of the form (6).

Since the mercury *does* evaporate, and since equations (1), (2), and (4) must be satisfied, our only alternative is the solution

$$\frac{du}{dx} = 0, \qquad \frac{d\rho}{dx} = 0, \qquad \frac{dp}{dx} = 0; \tag{7}$$

whence

$$\begin{aligned} u &= \text{constant}, \\ \rho &= \text{constant}, \\ p &= \text{constant}. \end{aligned} \tag{8}$$

From this we see that *in the steady state the pressure drop across the gas column is zero*; hence since the saturation vapor pressure of the hot liquid is considerably greater than that of the cold liquid, we see that the assumption that the pressure of the gas in the vicinity of the liquid is the saturation vapor pressure of the liquid leads to an immediate contradiction. It thus follows that the gas cannot be isotropic in the immediate vicinity of the liquid surface.

From the standpoint of kinetic theory the temperature of a gas is a measure of the random motion of the molecules, the gas being isotropic. We have seen that a gas evaporating from the surface of a liquid is not isotropic in the vicinity of the liquid; thus the concept of a single gas

temperature becomes meaningless; furthermore, any attempt to treat the problem of evaporation and retain this concept leads to a contradiction. If, however, we assign *two temperatures* to the gas, a "longitudinal temperature" in the direction of evaporation, and a "transverse temperature" at right angles to this direction; then the dependence of the random molecular motion upon direction is taken into account; and the problem of evaporation can be solved. In the sections which follow, this concept of two temperatures is applied to the problem of evaporation of monatomic gases; and the results are placed in the form of tables and curves suitable for practical use. Examples are given in Section 10 which illustrate the use of this data.

It is to be emphasised that the analysis which follows applies directly and without modification to the process of sublimation; thus *if throughout this paper we replace the word "evaporation" by "sublimation", and the word "liquid" by "solid", all methods and results apply in full to the sublimation of monatomic gases.*

Throughout this paper all units are C. G. S., and all temperatures are in degrees absolute, or Kelvin, except where it is expressly stated otherwise.

2. **Method of Attack. Fundamental Assumptions.** The fundamental assumption which is made concerning the mechanism of evaporation is that secondary emission is a negligible factor. It is thus assumed that the velocity distribution of the emitted molecules is not affected by that of the incident, or returning molecules. Thus in particular the velocity distribution of the emitted molecules is the same as in the case when the liquid is in equilibrium with its saturated vapor, in which case the gas is isotropic and homogeneous. The velocity distribution of the emitted molecules must thus be the same as that of the molecules crossing from one side of any plane in the saturated vapor to the other side. It thus must be

$$dN = \nu_0 \left(\frac{h_0 m}{\pi}\right)^{\frac{1}{2}} u\epsilon^{-h_0 m(u^2+v^2+w^2)}\, dudvdw\,, \qquad u \geqq 0\,, \tag{9}$$

as is shown in Section 3. Here u, v, w are the velocity components parallel to the x, y, and z axes, respectively, of a set of Cartesian coordinates placed with the origin at the liquid surface and the positive x direction normal to the surface outward. dN is the number of molecules emitted per second per square centimeter having u, v, and w components

of velocity lying between u and $u + du$, v and $v + dv$, and w and $w + dw$, respectively; furthermore

$$\begin{aligned} \nu_0 &= \text{molecular density of saturated vapor,} \\ m &= \text{mass of molecule,} \\ h_0 &= \text{temperature coefficient in Maxwellian velocity distribution of saturated vapor, } (h_0 = 3.65 \cdot 10.^{15} \div \text{absolute temperature}). \end{aligned} \tag{10}$$

It is evident that the relation (9) completely specifies the mechanical behavior of the liquid.

We shall next obtain certain equilibrium conditions which must be satisfied in the steady state by any gas regardless of whether or not it is isotropic and homogeneous. To this end let us consider that portion of a gas in the steady state which lies in any arbitrary but fixed closed surface, this surface being stationary. Because of steady state conditions it is clear that the total energy enclosed by this surface is constant, this energy being merely the translatory energy of the enclosed molecules, since the gas is monatomic. Since the total energy of two molecules is not affected by a collision of these molecules, we see that the molecular collisions which occur within the surface have no effect upon the total energy enclosed; hence the rate of increase of the total energy enclosed by the surface is the rate at which molecular energy enters the surface.[2] Since the total energy is constant, we thus see that the total molecular energy entering the surface per unit time is zero; thus the total energy of the molecules entering the surface per second is the same as the total energy of the molecules leaving the surface per second. The above argument applies directly to the mass and to the three components of linear momentum, as well as to the energy; thus we have the following

[2] It is evident that we might arbitrarily form the sum $\sum_{i=1}^{n_V} f(u_i, v_i, w_i)$ taken over all of the molecules enclosed by the surface, where f is any arbitrary function of the velocity components; then this sum must be constant because of steady state conditions. However, in the case of a collision of two molecules the sum of the corresponding two terms is in general altered abruptly; thus the total sum is affected by the molecular collisions within the surface; and it no longer follows that the sum $\sum_{i=1}^{n_A} f(u_i, v_i, w_i)$ taken over all molecules entering the surface per second is zero.

five equilibrium conditions which must be satisfied in the steady state for any fixed closed surface:

a) The total number of molecules entering the surface per second is zero.
b) The total x component of momentum entering the surface per second is zero.
c), *d*) Same as *b*) except for y and z components of momentum, respectively. (11)
e) The total molecular energy entering the surface per second is zero.

Obviously these are necessary conditions.

We now consider the one dimensional[3] problem of a monatomic gas evaporating from a liquid; thus placing a coördinate system as with (9), we apply the conditions (11) to the closed disc-shaped surface formed by the liquid surface, a parallel plane surface in the gas spaced a distance Δx from the liquid, and that portion of a right circular cylinder normal to these surfaces which is intercepted by them, the cross section of the cylinder being one square centimeter. We now introduce the following notation concerning gas flow across a unit plane area placed parallel to the liquid surface at a distance x from it:

M^+ = total mass of molecules crossing this area per second in the positive x direction,
M^- = total mass of molecules crossing this area per second in the negative x direction,
μ^+ = total x component of momentum of molecules crossing this area per second in the positive x direction, (12)
μ^- = total x component of momentum of molecules crossing this area per second in the negative x direction,
E^+ = total energy of molecules crossing this area per second in the positive x direction,
E^- = total energy of molecules crossing this area per second in the negative x direction.

[3] By "one dimensional" we merely mean that the state of the gas varies only in the direction normal to the liquid surface. We do not infer that there are no components of molecular velocities parallel to this surface; in fact such parallel components are included in (9).

Obviously these quantities are all functions of x, since the state of the gas depends upon x. For the liquid surface we also place

$$\begin{aligned} M_0^+ &= \text{total mass of molecules emitted per second per square centimeter,} \\ \mu_0^+ &= \text{total } x \text{ component of momentum of molecules emitted per second per square centimeter,} \\ E_0^+ &= \text{total energy of molecules emitted per second per square centimeter.} \end{aligned} \tag{13}$$

Obviously the quantities (13) are determined by (9).

Applying condition *a*) of (11) to the disc-shaped surface, noting that the edges of the disc may be neglected since the problem is one dimensional, we have

$$M_0^+ - (M^-)_{x=0} - (M^+)_{x=\Delta x} + (M^-)_{x=\Delta x} = 0. \tag{14}$$

But using the first law of the mean we see that

$$\begin{aligned} (M^+)_{x=\Delta x} &= (M^+)_{x=0} + \left(\frac{dM^+}{dx}\right)_{x=x_1} \Delta x \\ (M^-)_{x=\Delta x} &= (M^-)_{x=0} + \left(\frac{dM^-}{dx}\right)_{x=x_2} \Delta x, \end{aligned} \tag{15}$$

where x_1 and x_2 lie somewhere in the interval $0 < x < \Delta x$; hence substituting (15) in (14), we have

$$M_0^+ - (M^+)_{x=0} = \left\{\left(\frac{dM^+}{dx}\right)_{x=x_1} - \left(\frac{dM^-}{dx}\right)_{x=x_2}\right\} \Delta x. \tag{16}$$

Finally we let $\Delta x \to 0$, and thus obtain the relation

$$M_0^+ = (M^+)_{x=0} \tag{17}$$

subject only to the condition that the coefficient of Δx in (16) does not become infinite in the vicinity of the liquid surface. In a precisely similar manner we may apply conditions *b*) and *e*) to the disc-shaped surface; thus with (17) we have the equilibrium conditions

$$\begin{aligned} M_0^+ &= (M^+)_{x=0} \\ \mu_0^+ &= (\mu^+)_{x=0} \\ E_0^+ &= (E^+)_{x=0}, \end{aligned} \tag{18}$$

which must be satisfied by the gas at the liquid surface. These equations are subject only to the condition that quantities corresponding to

the coefficient of Δx in (16) remain finite in the vicinity of the liquid surface, which condition is obviously satisfied if the quantities

$$\frac{dM^+}{dx}, \quad \frac{dM^-}{dx}, \quad \frac{d\mu^+}{dx}, \quad \frac{d\mu^-}{dx}, \quad \frac{dE^+}{dx}, \quad \frac{dE^-}{dx} \tag{19}$$

remain finite in this vicinity. From the physical nature of the problem we see that the state of the gas cannot vary abruptly with x, and that the quantities M^+, M^-, $\cdots$, E^- are nowhere infinite; hence the quantities (19) must remain finite, and the above requirement is satisfied. The relations (18) are thus established. It may be noted that no use could be made of conditions *c*) and *d*) of (11) because of the one dimensional nature of the problem.

We do not know the velocity distribution of the molecules of the gas just outside of the liquid surface; however we do know that it must satisfy the equations (18); furthermore since all directions parallel to the liquid surface are equivalent, we see that the distribution function must be an even function of v and of w, so that the corresponding distribution curves are symmetrical about the origin. Our next step is to assume a reasonable form of distribution function containing a sufficient number of constants, or parameters, to render possible a good approximation to actual conditions. These constants are then determined so as to satisfy (18), and also the condition that the rate of evaporation take a specified value; thus making four conditions in all. The assumed form of distribution function must therefore contain four constants.

It is now at once evident that we cannot assume the distribution function corresponding to an isotropic gas having a drift velocity u_0 normal to the liquid surface; for although such a function is even in v and in w, it contains but three constants instead of four: drift velocity, temperature, and density. These three constants may be determined so that (18) is satisfied; but then we have no choice as to the rate of evaporation. In fact we see at once that the solution in this case is $u_0 = 0$, temperature = liquid temperature, density = density of stationary saturated vapor; hence it follows that the rate of evaporation is zero. The assumption of an isotropic gas moving with drift velocity u_0 thus leads merely to the trivial case of a stationary saturated vapor in equilibrium with its liquid.

Throughout the remainder of this paper we shall assume a distribution function of the form

$$Ae^{-m[\lambda_2(u-u_0)^2+\lambda_3 v^2+\lambda_4 w^2]} \tag{20}$$

where A, h_L, h_T, u_0 are constants. Since the molecular density ν is given by the expression

$$\nu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\epsilon^{-m[h_L(u-u_0)^2+h_T v^2+h_T w^2]}\,du\,dv\,dw$$

it follows by direct integration that

$$\nu = A\left(\frac{\pi^3}{h_L h_T^2 m^3}\right)^{\frac{1}{2}};$$

whence (20) becomes

$$\nu\left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \epsilon^{-m[h_L(u-u_0)^2+h_T v^2+h_T w^2]}. \tag{21}$$

This expression multiplied by *dudvdw* gives the number of molecules per cubic centimeter of the gas at the liquid surface which have velocity components lying between u and $u + du$, v and $v + dv$, w and $w + dw$, respectively. The expression (21) differs from a Maxwellian distribution[4] in that the u velocity component has a temperature coefficient, h_L, which differs from that of the other two components. The four constants which fix the nature of the gas are

$$\begin{aligned} \nu &= \text{molecular density},\\ h_L &= \text{longitudinal temperature coefficient},\\ h_T &= \text{transverse temperature coefficient},\\ u_0 &= \text{drift velocity}. \end{aligned} \tag{22}$$

As in the case of an ordinary isotropic gas we *define*

$$\begin{aligned} T_L &= \frac{1}{2h_L k} = \text{longitudinal temperature},\\ T_T &= \frac{1}{2h_T k} = \text{transverse temperature},\\ k &= \text{Boltzman's constant} = 1.371\cdot 10.^{-16}. \end{aligned} \tag{23}$$

The constants (22) are determined by (18) and the condition that the rate of evaporation take a specified value; thus the nature of the gas *at*

[4] Jeans, "Dynamical Theory of Gases", p. 33.
Loeb, "Kinetic Theory of Gases", p. 59.
Schaefer, "Einführung in die theoretische Physik", vol. 2, p. 339.

the liquid surface is completely determined subject to the assumption (20).

As the gas moves away from the surface, the molecules collide with each other, and in the course of several mean free paths again effect an equipartition of energy; so that the gas in its final state is again isotropic, and thus has the Maxwellian distribution function

$$\nu_1 \left(\frac{h_1 m}{\pi}\right)^{\frac{3}{2}} \epsilon^{-h_1 m[(u-u_1)^2+v^2+w^2]}, \tag{24}$$

which is determined by the three constants

$$\begin{aligned} \nu_1 &= \text{molecular density}, \\ h_1 &= \text{temperature coefficient}, \\ u_1 &= \text{drift velocity}. \end{aligned} \tag{25}$$

Here again we have by definition

$$T_1 = \frac{1}{2h_1 k} = \text{temperature}. \tag{26}$$

The constants (25) are determined by applying conditions *a*), *b*), and *e*), of (11) to the closed surface bounded by a cylinder of unit cross section normal to the liquid surface and by two planes parallel to this surface: one in the gas just outside, and the other spaced far enough away so that it lies in gas which has reached its final isotropic state. Thus the final state of the gas for a given rate of evaporation is determined.

3. **Transfer of Mass, Momentum, and Energy Across Unit Area Normal to the Drift Velocity.** We now proceed with the calculation of the quantities (12) for the distribution function (21). Since we are concerned only with transfer across unit area of a plane perpendicular to the x axis, it will be convenient to have the distribution function for the u component of velocity alone; thus integrating (21) with respect to v and w, we obtain the function

$$\begin{aligned} \nu \left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon^{-m[h_L(u-u_0)^2+h_T v^2+h_T w^2]}\, dv\, dw \\ = \nu \left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}} \epsilon^{-h_L m(u-u_0)^2}. \end{aligned} \tag{27}$$

This expression multiplied by du gives the molecular density of molecules having a u component of velocity lying between u and $u + du$.

In the derivations which are to come, the following integration formulas are useful:[5]

$$\int_{-u_0}^{\infty} x^2\epsilon^{-hmx^2}\,dx = \frac{u_0^2\epsilon^{-hmu_0^2}}{2hm} + \frac{\epsilon^{-hmu_0^2}}{2(hm)^2},$$

$$\int_{-u_0}^{\infty} x^2\epsilon^{-hmx^2}\,dx = \frac{-u_0\epsilon^{-hmu_0^2}}{2hm} + \frac{1}{4hm}\left(\frac{\pi}{hm}\right)^{\frac{1}{2}} + \frac{1}{2(hm)^{\frac{1}{2}}}\int_0^{u_0\sqrt{hm}} \epsilon^{-x^2}\,dx,$$

$$\int_{-u_0}^{\infty} x\epsilon^{-hmx^2}\,dx = \frac{\epsilon^{-hmu_0^2}}{2hm}, \tag{28}$$

$$\int_{-u_0}^{\infty} \epsilon^{-hmx^2}\,dx = \frac{1}{2}\left(\frac{\pi}{hm}\right)^{\frac{1}{2}} + \frac{1}{\sqrt{hm}}\int_0^{u_0\sqrt{hm}} \epsilon^{-x^2}\,dx.$$

These are readily obtained by successive integration by parts.

Transfer of Mass. We shall first determine $(M^+)_{x=0}$, the subscript merely referring to a gas in the state (21); thus multiplying the velocity u by the molecular density (27) and integrating, we have

$$(M^+)_{x=0} = m\nu\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} u\epsilon^{-h_L m(u-u_0)^2}\,du. \tag{29}$$

Placing $x = u - u_0$, (29) becomes[5]

$$(M^+)_{x=0} = m\nu\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}} \int_{-u_0}^{\infty} (x + u_0)\,\epsilon^{-h_L m x^2}\,dx;$$

whence noting (28) we have finally

$$(M^+)_{x=0} = m\nu\left\{\frac{\epsilon^{-h_L m u_0^2}}{2\sqrt{\pi h_L m}} + \frac{u_0}{2}\left(1 + \frac{2}{\sqrt{\pi}}\int_0^{u_0\sqrt{h_L m}} \epsilon^{-x^2}\,dx\right)\right\}. \tag{30}$$

Similarly, for $(M^-)_{x=0}$ we have

$$(M^-)_{x=0} = -m\nu\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{0} u\epsilon^{-h_L m(u-u_0)^2}\,du;$$

whence placing $u = -\xi$ we obtain an expression identical with (29) except that the sign of u_0 in the exponent is plus instead of minus. We

[5] x is used here merely as an auxiliary integration variable, and does not refer to the coördinate system of Section 2.

may thus obtain $(M^-)_{x=0}$ by reversing the sign of u_0 in (30); thus we have finally

$$(M^-)_{x=0} = m\nu\left\{\frac{\epsilon^{-h_L m u_0^2}}{2\sqrt{\pi h_L m}} - \frac{u_0}{2}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{u_0\sqrt{h_L m}} \epsilon^{-x^2}\,dx\right)\right\}. \tag{31}$$

From (30) and (31) we see that

$$(M^+)_{x=0} - (M^-)_{x=0} = m\nu u_0,$$

as was to be expected.

Transfer of Momentum. We shall next determine $(\mu^+)_{x=0}$, the subscript referring to a gas in the state (21); thus proceeding as with (29), we have

$$(\mu^+)_{x=0} = m\nu\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}}\int_0^\infty u^2\epsilon^{-h_L m(u-u_0)^2}\,du. \tag{32}$$

Placing $x = u - u_0$, (32) becomes[5]

$$(\mu^+)_{x=0} = m\nu\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}}\int_{-u_0}^\infty (x^2 + 2u_0 x + u_0^2)\,\epsilon^{-h_L m x^2}\,dx;$$

whence, noting (28), we have finally

$$(\mu^+)_{x=0} = \frac{m\nu}{2}\left\{\frac{u_0\epsilon^{-h_L m u_0^2}}{\sqrt{\pi h_L m}} + \left(u_0^2 + \frac{1}{2h_L m}\right) \times\left(1 + \frac{2}{\sqrt{\pi}}\int_0^{u_0\sqrt{h_L m}} \epsilon^{-x^2}\,dx\right)\right\}. \tag{33}$$

Similarly, for $(\mu^-)_{x=0}$ we have

$$(\mu^-)_{x=0} = -m\nu\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^0 u^2\epsilon^{-h_L m(u-u_0)^2}\,du;$$

whence placing $u = -\xi$, we obtain an expression which differs from (32) only in the sign of u_0, and in the initial minus sign. By reversing the sign of u_0 in (33) and prefixing a minus sign, we thus have finally

$$(\mu^-)_{x=0} = \frac{m\nu}{2}\left\{\frac{u_0\epsilon^{-h_L m u_0^2}}{\sqrt{\pi h_L m}} - \left(u_0^2 + \frac{1}{2h_L m}\right) \times\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{u_0\sqrt{h_L m}} \epsilon^{-x^2}\,dx\right)\right\}. \tag{34}$$

Transfer of Energy. We now proceed to determine $(E^+)_{x=0}$, the subscript referring to a gas in the state (21). Multiplying the molecular

density (21) by u, we obtain the number of molecules having velocity components lying between u and $u + du$, v and $v + dv$, w and $w + dw$, respectively, which cross unit area of a plane perpendicular to the x axis per second in the positive direction; thus multiplying by $\frac{1}{2}m(u^2 + v^2 + w^2)$, the energy per molecule, and integrating, we have

$$(E^+)_{x=0} = \frac{m\nu}{2}\left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty u(u^2 + v^2 + w^2) \times \epsilon^{-m[h_L(u-u_0)^2 + h_T v^2 + h_T w^2]}\, du\, dv\, dw \tag{35}$$

$$= \frac{m\nu}{2}\left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \int_0^\infty u\epsilon^{-h_L m(u-u_0)^2} \left\{u^2 \left(\int_{-\infty}^\infty \epsilon^{-h_T m v^2}\, dv\right) \times \left(\int_{-\infty}^\infty \epsilon^{-h_T m w^2}\, dw\right) + \left(\int_{-\infty}^\infty \epsilon^{-h_T m w^2}\, dw\right)\left(\int_{-\infty}^\infty v^2 \epsilon^{-h_T m v^2}\, dv\right) + \left(\int_{-\infty}^\infty \epsilon^{-h_T m v^2}\, dv\right)\left(\int_{-\infty}^\infty w^2 \epsilon^{-h_T m w^2}\, dw\right)\right\} du\,.$$

Noting (28) we evaluate the v and w integrals; thus we have

$$(E^+)_{x=0} = \frac{m\nu}{2}\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}} \int_0^\infty u\left(u^2 + \frac{1}{h_T m}\right)\epsilon^{-h_L m(u-u_0)^2}\, du\,. \tag{36}$$

Placing $x = u - u_0$, (36) becomes[5]

$$(E^+)_{x=0} = \frac{m\nu}{2}\left(\frac{h_L m}{\pi}\right)^{\frac{1}{2}} \int_{-u_0}^\infty \left[x^3 + 3u_0 x^2 + \left(3u_0^2 + \frac{1}{h_T m}\right)x + \left(u_0^3 + \frac{u_0}{h_T m}\right)\right]\epsilon^{-h_L m x^2}\, dx\,,$$

an integral which can be evaluated by means of (28); thus we have finally

$$(E^+)_{x=0} = \frac{m\nu}{2}\left\{\left(u_0^2 + \frac{1}{h_L m} + \frac{1}{h_T m}\right)\frac{\epsilon^{-h_L m u_0^2}}{2\sqrt{\pi h_L m}} + \tfrac{1}{2}\left(u_0^3 + \frac{3u_0}{2h_L m} + \frac{u_0}{h_T m}\right)\left(1 + \frac{2}{\sqrt{\pi}}\int_0^{u_0\sqrt{h_L m}} \epsilon^{-x^2}\, dx\right)\right\}. \tag{37}$$

Proceeding as with (35) we obtain the following expression for $(E^-)_{x=0}$:

$$(E^-)_{x=0} = -\frac{m\nu}{2}\left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \int_{-\infty}^0 \int_{-\infty}^\infty \int_{-\infty}^\infty u(u^2 + v^2 + w^2) \times \epsilon^{-m[h_L(u-u_0)^2 + h_T v^2 + h_T w^2]}\, du\, dv\, dw\,.$$

Placing $u = -\xi$, we obtain an expression that differs from (35) only in that the sign of u_0 is plus instead of minus. We may thus obtain $(E^-)_{x=0}$ by reversing the sign of u_0 in (37); thus we have finally

$$(E^-)_{x=0} = \frac{m\nu}{2}\left\{\left(u_0^2 + \frac{1}{h_L m} + \frac{1}{h_T m}\right)\frac{\epsilon^{-h_L m u_0^2}}{2\sqrt{\pi h_L m}} - \frac{1}{2}\left(u_0^2 + \frac{3u_0}{2h_L m} + \frac{u_0}{h_T m}\right)\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{u_0\sqrt{h_L m}} \epsilon^{-x^2}\,dx\right)\right\}. \tag{38}$$

Simplification and Summary of Transfer Equations. The use of the above equations of transfer is greatly simplified if we place

$$\rho = u_0\sqrt{h_L m}, \qquad \sigma = \frac{h_L}{h_T}, \qquad P(\rho) = \frac{2}{\sqrt{\pi}}\int_0^{\rho} \epsilon^{-x^2}\,dx = \text{Probability Integral}; \tag{39}$$

thus from (30), (31), (33), (34), (37), (38), we have

$$\begin{aligned}
(M^+)_{x=0} &= \frac{m\nu}{2\sqrt{h_L m}}\left\{\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\,[1 + P(\rho)]\right\}\\
(M^-)_{x=0} &= \frac{m\nu}{2\sqrt{h_L m}}\left\{\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho\,[1 - P(\rho)]\right\}\\
(\mu^+)_{x=0} &= \frac{m\nu}{2h_L m}\left\{\frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} + \left[\rho^2 + \frac{1}{2}\right][1 + P(\rho)]\right\}\\
(\mu^-)_{x=0} &= \frac{m\nu}{2h_L m}\left\{\frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} - \left[\rho^2 + \frac{1}{2}\right][1 - P(\rho)]\right\}\\
(E^+)_{x=0} &= \frac{m\nu}{4(h_L m)^{\frac{1}{2}}}\left\{[\rho^2 + 1 + \sigma]\,\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)]\right\}\\
(E^-)_{x=0} &= \frac{m\nu}{4(h_L m)^{\frac{1}{2}}}\left\{[\rho^2 + 1 + \sigma]\,\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 - P(\rho)]\right\}.
\end{aligned} \tag{40}$$

The simplification lies in the reduction of the number of variables contained in the braces.

We still have the problem of determining the quantities (13), which problem is that of determining the transfer equations for the liquid;

however, comparing (9) and (21), we see that the quantities (13) can be written down immediately by placing $\rho = 0$, $\sigma = 1$, $h_L = h_0$, and $\nu = \nu_0$ in (40). We thus have finally

$$M_0^+ = \frac{m\nu_0}{2\sqrt{\pi h_0 m}}$$

$$\mu_0^+ = \frac{m\nu_0}{4h_0 m} \tag{41}$$

$$E_0^+ = \frac{m\nu_0}{2\sqrt{\pi}(h_0 m)^{\frac{1}{2}}}.$$

4. **Equilibrium Conditions—Liquid to Anisotropic Gas.** Now that we have the transfer equations (40) and (41) for the anisotropic gas and the liquid, respectively, we can apply the equilibrium conditions (18); thus substituting, we have

$$\frac{m\nu_0}{2\sqrt{\pi h_0 m}} = \frac{m\nu}{2\sqrt{h_L m}}\left\{\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho[1 + P(\rho)]\right\}$$

$$\frac{m\nu_0}{4h_0 m} = \frac{m\nu}{2h_L m}\left\{\frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} + \left[\rho^2 + \frac{1}{2}\right][1 + P(\rho)]\right\} \tag{42}$$

$$\frac{m\nu_0}{2\sqrt{\pi}(h_0 m)^{\frac{1}{2}}} = \frac{m\nu}{4(h_L m)^{\frac{1}{2}}}\left\{[\rho^2 + 1 + \sigma]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)]\right\}.$$

We now place

$$\frac{\nu}{\nu_0} = \nu',$$

$$\frac{h_L}{h_0} = h_L',$$

$$\phi_1(\rho) = \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho[1 + P(\rho)], \tag{43}$$

$$\phi_2(\rho) = \frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} + \left[\rho^2 + \frac{1}{2}\right][1 + P(\rho)],$$

$$\phi_3(\rho, \sigma) = [\rho^2 + 1 + \sigma]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)];$$

whence substituting, (42) becomes

$$\frac{\nu'\sqrt{\pi}}{\sqrt{h_L'}}\phi_1 = 1$$

$$\frac{2\nu'}{h_L'}\phi_2 = 1 \tag{44}$$

$$\frac{\nu'\sqrt{\pi}}{2h_L'^{\frac{1}{2}}}\phi_3 = 1.$$

We now solve the equations (44) to obtain σ, h_L', and ν' in terms of ρ; thus dividing the last two equations of (44) by the first, we eliminate ν'; whence

$$\frac{2}{\sqrt{\pi h_L'}}\left(\frac{\phi_2}{\phi_1}\right) = 1$$

$$\frac{1}{2h_L'}\left(\frac{\phi_3}{\phi_1}\right) = 1. \tag{45}$$

Dividing the second equation of (45) by the first squared, we eliminate h_L'; thus we have

$$\frac{\pi}{8}\left(\frac{\phi_1\phi_3}{\phi_2^2}\right) = 1. \tag{46}$$

Solving (46) for σ, noting (43), it follows that

$$\sigma = \frac{1}{\phi_1}\left\{\frac{8\phi_2^2}{\pi\phi_1} - [\rho^2 + 1]\frac{e^{-\rho^2}}{\sqrt{\pi}} - \rho\left[\rho^2 + \frac{3}{2}\right][1 + P(\rho)]\right\}, \tag{47}$$

an expression giving σ explicitly in terms of ρ.

From the first equation of (45) we have at once

$$h_L' = \frac{4}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2. \tag{48}$$

From (23) and (39) we also have

$$h_T' = \frac{h_L'}{\sigma}$$

$$\frac{T_L}{T_0} = \frac{1}{h_L'} \tag{49}$$

$$\frac{T_T}{T_0} = \frac{\sigma}{h_L'}.$$

From (48) and the first equation of (44) we now have

$$\nu' = \frac{2}{\pi}\left(\frac{\phi_2}{\phi_1^2}\right). \tag{50}$$

The rate of evaporation per unit area of liquid is obviously

$$M = (M^+)_{x=0} - (M^-)_{x=0} = m\nu u_0 = \frac{m\nu\rho}{\sqrt{h_L m}};$$

whence it follows from (41), (48), and (50) that

$$\frac{M}{M_0^+} = \frac{2\rho}{\phi_1}. \tag{51}$$

We finally obtain the pressure p_s acting on the liquid surface; thus from the second equation of (18) we have at once

$$p_s = \mu_0^+ - (\mu^-)_{x=0} = (\mu^+)_{x=0} - (\mu^-)_{x=0};$$

whence substituting from (40) it follows that

$$p_s = \frac{\nu}{h_L}\left(\rho^2 + \frac{1}{2}\right). \tag{52}$$

The saturation vapor pressure may be obtained from (52) by placing $\nu = \nu_0$, $h_L = h_0$, $\rho = 0$; thus

$$p_0 = \frac{\nu_0}{2h_0}. \tag{53}$$

Dividing (52) by (53) noting the second equation of (44), we have finally

$$\frac{p_s}{p_0} = \frac{1}{\phi_2}\left(\rho^2 + \frac{1}{2}\right). \tag{54}$$

If desired, the drift velocity may be obtained from (39). We see that equations (47), (48), (49), (50), (51), and (54) give the various constants of the anisotropic gas in terms of those of the saturated vapor, and the single parameter ρ.

5. **Calculations, Tables, and Curves for the Anisotropic Gas.** We have seen that in Section 4 all quantities are finally expressed in terms of the parameter ρ, the value $\rho = 0$ corresponding to the stationary saturated vapor; however we have no a priori means of knowing what set of values of ρ to consider. Because of this uncertainty a set of

preliminary calculations was carried through in which ρ was arbitrarily given the values 0, .2, .4, .6, $\cdots$, 2.0; then on the basis of these calculations, and because, as we shall see later, we are primarily interested in values of ρ smaller than .51, the set of values of ρ given in Table I was chosen as a basis for final calculations. Values of $\frac{2}{\sqrt{\pi}}\int_0^\rho e^{-x^2}\,dx$ and $\frac{2\epsilon^{-\rho^2}}{\sqrt{\pi}}$ were taken from a paper by Burgess.[6] The calculations were carried out on a Marchant computing machine capable of handling nine place factors. All machine calculations were carried through three times, for each of which the recorded value was checked three times. The calculations themselves consisted merely in carrying out the operations indicated in Section 4. The results are given in Table I, and in Figs. 1, 2, 3, 4, 5, and 6. The curves Figs. 5 and 6 are obtained by plotting $\frac{p_s}{p_0}$, $\frac{T_L}{T_0}$, and $\frac{T_T}{T_0}$, respectively, against $\frac{M}{M_0^+}$ for the same values of ρ, thus eliminating ρ. All figures tabulated are significant.[7]

From the curves we see clearly how as the rate of evaporation and hence ρ are increased from zero, the state of the gas at the liquid surface leaves that of the stationary saturated vapor, the transverse temperature becoming somewhat higher, the longitudinal temperature becoming considerably lower than the liquid temperature. We also see how the pressure on the liquid surface falls off as the rate of evaporation is increased.

6. **Partition of Energy in the Anisotropic Gas. Significance of Two Temperatures.** For an ordinary isotropic gas we know that the temperature is a measure of the mean kinetic energy per molecule. The temperatures T_L and T_T of an anisotropic gas were defined by (23) in terms of the constants h_L and h_T, which occur in the distribution function (21); thus they have as yet no significance in terms of mean molec-

[6] Burgess, "On the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt$, with Extended Tables of Values", *Transactions of the Royal Society of Edinburgh*, vol. 39, p. 257.

[7] By this statement we mean that the difference between a value tabulated and the true value amounts to less than one unit in the last decimal place. That such a statement can be made follows from the fact that a set of calculations involving differences was made paralleling the machine calculations here described, by means of which bounds upon the error were obtained at each step.

ular energy. To determine any such significance let us compute the mean kinetic energy per molecule of the anisotropic gas; thus noting (21), we see that this is given by the expression

$$\frac{m}{2}\left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u^2+v^2+w^2)\, \epsilon^{-m[h_L(u-u_0)^2+h_T v^2+h_T w^2]} \times du\, dv\, dw\,. \qquad (55)$$

The integral (55) can be evaluated by the aid of (28) in essentially the same manner as (35); thus the average energy per molecule of a gas in the state (21) is found to be

$$\frac{1}{2}mu_0^2 + \frac{1}{4h_L} + \frac{1}{2h_T},$$

or

$$\frac{1}{2}mu_0^2 + \frac{1}{2}kT_L + kT_T\,. \qquad (56)$$

From (56) we see that the mean molecular energy may be considered as made up of the drift energy $\frac{1}{2}mu_0^2$, the thermal energy $\frac{1}{2}kT_L$ for the x coördinate, or degree of freedom, and the thermal energies $\frac{1}{2}kT_T$ each for the y and z coördinates. We thus see that *the transverse and longitudinal temperatures are a direct measure of the mean molecular energy for the corresponding coördinates, or degrees of freedom, respectively*; hence these temperatures have the same significance as has the single temperature of an isotropic gas. We see from (56) that we obtain an equipartition of energy only in the particular case where $T_L = T_T = T$, in which case the gas is isotropic.

Referring to Fig. 6, we see that for a given rate of evaporation the percentage deviation of T_T from T_0 is small, and furthermore is much smaller than is that of T_L; hence noting (56) we infer that the three degrees of freedom tend to remain isolated from each other during the process of evaporation. Thus in a sense drift energy is obtained at the expense of the thermal energy of the x coördinate.[8]

[8] As the gas moves away from the liquid surface its state changes continuously, and approaches the isotropic; however it may at any point be specified by the four constants T_L, T_T, ν, u_0, which must thus be looked upon as functions of x. Condition e) of (11) applied to the closed surface made up of the liquid surface, a plane parallel to this surface at a distance x from it, and a right circular cylinder of unit cross section normal to it requires that the quantity $[E^+ - E^-]$ remain

7. Equilibrium Conditions—Anisotropic to Isotropic Gas. Having determined the state of the gas at the liquid surface, we now proceed to determine its state at a distance of several mean free paths from the surface, where the gas is again isotropic. We shall thus proceed to determine the constants (25) of the distribution (24) in the manner described in Section 2. Let us denote by x_1 the abscissa of the right hand boundary of the closed surface, which boundary is the plane lying in the isotropic gas; then noting (12) we see that conditions a, b, and e, of (11) may be written

$$\begin{aligned}(M^+)_{x=x_1} - (M^-)_{x=x_1} &= (M^+)_{x=0} - (M^-)_{x=0}\\ (\mu^+)_{x=x_1} - (\mu^-)_{x=x_1} &= (\mu^+)_{x=0} - (\mu^-)_{x=0}\\ (E^+)_{x=x_1} - (E^-)_{x=x_1} &= (E^+)_{x=0} - (E^-)_{x=0}\,.\end{aligned} \tag{57}$$

The right hand sides of these equations follow at once from (40); thus

$$\begin{aligned}(M^+)_{x=0} - (M^-)_{x=0} &= \frac{m\nu\rho}{\sqrt{h_L m}}\\ (\mu^+)_{x=0} - (\mu^-)_{x=0} &= \frac{m\nu}{h_L m}\left(\rho^2 + \frac{1}{2}\right)\\ (E^+)_{x=0} - (E^-)_{x=0} &= \frac{m\nu\rho}{2(h_L m)^{3/2}}\left(\rho^2 + \frac{3}{2} + \sigma\right).\end{aligned} \tag{58}$$

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constant for all values of x, this quantity being the energy transfer per unit area. Referring to (37) and (38) we see that

$$[E^+ - E^-] = \nu u_0\left[\frac{1}{2}mu_0^2 + \frac{3}{4h_L} + \frac{1}{2h_T}\right];$$

hence since νu_0 is independent of x, being the net number of molecules crossing unit area per second, it follows that the quantity $\left[\frac{1}{2}mu_0^2 + \frac{3}{4h_L} + \frac{1}{2h_T}\right]$ must remain constant. Denoting the mean molecular energy by e, and noting (23) and (56), we see that the quantity $[e + kT_L]$ remains constant for all values of x; also we see that the quantity e is not constant, but depends upon x.

Throughout this paper with the one exception of this footnote the symbols T_L, T_T, ν, u_0, h_L, h_T refer to the anisotropic gas *at the liquid surface*, and *not* to the gas at a distance x from this surface.

Comparing (24) with (21) we see that the left hand members of the equations (57) may be obtained by placing $x = x_1$, $\nu = \nu_1$, $h_L = h_1$, and $\sigma = 1$ in (58); and by writing

$$\rho_1 = u_1 \sqrt{h_1 m} \tag{59}$$

in place of ρ. It thus follows that

$$\begin{aligned}
(M^+)_{x=x_1} - (M^-)_{x=x_1} &= \frac{m\nu_1\rho_1}{\sqrt{h_1 m}} \\
(\mu^+)_{x=x_1} - (\mu^-)_{x=x_1} &= \frac{m\nu_1}{h_1 m}\left(\rho_1^2 + \frac{1}{2}\right) \\
(E^+)_{x=x_1} - (E^-)_{x=x_1} &= \frac{m\nu_1\rho_1}{2(h_1 m)^{3/2}}\left(\rho_1^2 + \frac{5}{2}\right).
\end{aligned} \tag{60}$$

Substituting (58) and (60) in (57), we have

$$\begin{aligned}
\frac{m\nu_1\rho_1}{\sqrt{h_1 m}} &= \frac{m\nu\rho}{\sqrt{h_L m}} \\
\frac{m\nu_1}{h_1 m}\left(\rho_1^2 + \frac{1}{2}\right) &= \frac{m\nu}{h_L m}\left(\rho^2 + \frac{1}{2}\right) \\
\frac{m\nu_1\rho_1}{2(h_1 m)^{3/2}}\left(\rho_1^2 + \frac{5}{2}\right) &= \frac{m\nu\rho}{2(h_L m)^{3/2}}\left(\rho^2 + \frac{3}{2} + \sigma\right);
\end{aligned} \tag{61}$$

whence placing

$$\begin{aligned}
\nu_1' &= \frac{\nu_1}{\nu}, \\
h_1' &= \frac{h_1}{h_L},
\end{aligned} \tag{62}$$

we finally obtain the equilibrium conditions

$$\begin{aligned}
\frac{\nu_1'\rho_1}{\sqrt{h_1'}} &= \rho \\
\frac{\nu_1'}{h_1'}\left(\rho_1^2 + \frac{1}{2}\right) &= \left(\rho^2 + \frac{1}{2}\right) \\
\frac{\nu_1'\rho_1}{h_1'^{3/2}}\left(\rho_1^2 + \frac{5}{2}\right) &= \rho\left(\rho^2 + \frac{3}{2} + \sigma\right).
\end{aligned} \tag{63}$$

We next proceed to solve the equations (63) to obtain the various constants of the isotropic gas in terms of ρ; thus dividing the last two equations by the first, we have

$$\frac{1}{\sqrt{h_1'}}\left(\frac{\rho_1^2+\frac{1}{2}}{\rho_1}\right)=\left(\frac{\rho^2+\frac{1}{2}}{\rho}\right)$$
$$\frac{1}{h_1'}\left(\rho_1^2+\frac{5}{2}\right)=\left(\rho^2+\frac{3}{2}+\sigma\right). \tag{64}$$

Dividing the second equation of (64) by the first squared, we have

$$\frac{\rho_1^2\left(\rho_1^2+\frac{5}{2}\right)}{\left(\rho_1^2+\frac{1}{2}\right)^2}=\frac{\rho^2\left(\rho^2+\frac{3}{2}+\sigma\right)}{\left(\rho^2+\frac{1}{2}\right)^2}. \tag{65}$$

Noting (43) and (47) we see that the right hand side of (65) involves only ρ; thus placing

$$\alpha=\frac{\rho^2\left(\rho^2+\frac{3}{2}+\sigma\right)}{\left(\rho^2+\frac{1}{2}\right)^2}, \tag{66}$$

and substituting, (65) becomes

$$\rho_1^2\left(\rho_1^2+\frac{5}{2}\right)=\alpha\left(\rho_1^2+\frac{1}{2}\right)^2,$$

or

$$(1-\alpha)\,\rho_1^4+\left(\frac{5}{2}-\alpha\right)\rho_1^2-\frac{1}{4}\,\alpha=0. \tag{67}$$

This expression is a biquadratic, the solution of which is

$$\rho_1^2=\frac{\alpha-\frac{5}{2}\pm\sqrt{\frac{25}{4}-4\alpha}}{2(1-\alpha)}, \tag{68}$$

an expression which gives ρ_1 explicitly in terms of ρ.

From the second equation of (64) we obtain h_1'; thus

$$h_1' = \frac{\rho_1^2 + \frac{5}{2}}{\rho^2 + \frac{3}{2} + \sigma}, \tag{69}$$

which expression together with (48) and (68) gives us $\frac{h_1}{h_0}$ in terms of ρ, for

$$\frac{h_1}{h_0} = h_1' h_L' . \tag{70}$$

Next we obtain $\frac{p_1}{p_0}$; thus dividing the relations

$$p_1 = \nu_1 k T_1$$

$$p_0 = \nu_0 k T_0 ,$$

noting (23) and (26), we have

$$\frac{p_1}{p_0} = \frac{\nu_1 h_0}{\nu_0 h_1} = \frac{\nu' \nu_1'}{h_L' h_1'} ; \tag{71}$$

whence from (48), (50), and the second equation of (63) it follows that

$$\frac{p_1}{p_0} = \frac{1}{2\phi_2} \left(\frac{\rho^2 + \frac{1}{2}}{\rho_1^2 + \frac{1}{2}} \right). \tag{72}$$

The relation (71) may be written

$$\frac{\nu_1}{\nu_0} = \left(\frac{h_1}{h_0} \right) \left(\frac{p_1}{p_0} \right), \tag{73}$$

an expression giving $\frac{\nu_1}{\nu_0}$ in terms of ρ by means of (70) and (72).

The temperature ratio

$$\frac{T_1}{T_0} = \frac{h_0}{h_1} \tag{74}$$

may at once be obtained from (70). The drift velocity u_1 may, if desired, be obtained from (59), (68), and (70). We have now obtained the various constants of the isotropic gas in terms of ρ.

8. Physical Significance of ρ and ρ_1. Limiting Value of Drift Velocity. The quantities ρ and ρ_1, defined by (39) and (59), respectively, were introduced merely to simplify the transfer equations (40) and (60), and were given no physical significance; however, we may write

$$\rho^2 = \frac{1}{2}\left(\frac{\frac{mu_0^2}{2}}{\frac{1}{4h_L}}\right)$$
$$\rho_1^2 = \frac{3}{2}\left(\frac{\frac{mu_1^2}{2}}{\frac{3}{4h_1}}\right). \tag{75}$$

Noting (56) we thus see that ρ^2 is one half the ratio of the drift energy per molecule to the mean longitudinal thermal energy per molecule of the anisotropic gas in the state (21). Similarly we see that ρ_1^2 is three halves the ratio of the drift energy per molecule to the mean thermal energy per molecule of the isotropic gas in the state (24).

An additional significance may be given to ρ_1 by noting that the velocity of sound in the isotropic gas is

$$u_s = \sqrt{\frac{5}{6h_1 m}}, \tag{76}$$

since the ratio of the specific heats of a monatomic gas is 5/3; thus noting (59) we have

$$\rho_1 = \frac{u_1}{u_s}\sqrt{\frac{5}{6}}. \tag{77}$$

We thus see that ρ_1 *is* $\sqrt{\frac{5}{6}}$ *times the ratio of the drift velocity to the velocity of sound in the isotropic gas.*

Referring to (68) we see that for there to be a real solution the expression under the radical must be positive; thus

$$\alpha \leqq \frac{25}{16}. \tag{78}$$

We thus see that values of ρ are restricted to those corresponding to values of α which satisfy (78); thus referring to Table I we see that

$$0 \leqq \rho \leqq .508047 \cdots. \tag{79}$$

When α takes its greatest allowable value, $\frac{25}{16}$, (68) becomes

$$\rho_1 = \sqrt{\frac{5}{6}}, \tag{80}$$

which in (77) gives

$$u_1 = u_s. \tag{81}$$

We thus see that *if an evaporating gas moves through a distance sufficient to allow the gas to become isotropic*[9] *then* ρ *is restricted to the range* (79), *the lower bound corresponding to the stationary saturated vapor; the upper bound corresponding to a drift velocity of the isotropic gas equal to the velocity of sound in that gas.* Referring to Table I we see that the rate of evaporation increases with ρ; whence it follows that *the maximum rate of evaporation occurs when the isotropic gas is moving with its sound velocity.*

Of the four values of ρ_1 defined by (68) for a given value of ρ, or α, we are obviously interested only in real positive values. These values are shown plotted against α in Fig 7. It may be noted that for the plus sign in (68) both the numerator and denominator vanish for $\alpha = 1$; however, applying L'Hospital's rule for indeterminate forms, we find that $\rho_1 \to \sqrt{\frac{1}{6}}$ as $\alpha \to 1$. We see that this curve has a vertical tangent for the critical value $\alpha = \frac{25}{16}$. The rate of evaporation is an increasing function of ρ; also we shall see (Table II) that the pressure p_1 is a decreasing function of ρ and of ρ_1. It thus follows that the portion of the curve lying above the critical point corresponds to a state of affairs such that a decrease in pressure causes a decrease in the rate of evaporation, which is impossible. This portion of the curve has no physical significance[10]—if the discharge pressure is reduced to a value below that corresponding to the critical point, the rate of evaporation retains its critical value. We thus see that ρ_1 takes its greatest possible value at

[9] As an example of a case where this condition is not satisfied, we could consider a condensing surface spaced a distance of less than several mean free paths from the evaporating liquid.

[10] This behavior is closely analogous to that encountered in the case of a gas discharging through an orifice from a region of high pressure to one of lower pressure.

Goodenough, "Principles of Thermodynamics", p. 146.

"Handbuch der Experimentalphysik", vol. 4, part 1, p. 387.

the critical point; whence it follows from (77) and (81) that *the drift velocity of the isotropic gas can never exceed its sound velocity.*

These results involving the velocity of sound may be shown to be independent of the form of the velocity distribution function assumed for the gas at the liquid surface (in this paper the function (21)), and may be shown to depend only upon (24), which is known to be exact, since the gas is isotropic at the point $x = x_1$. Thus let us assume in place of (21) an arbitrary distribution

$$f(u, v, w, \beta_1, \beta_2, \cdots, \beta_n), \tag{82}$$

where $\beta_1, \beta_2, \cdots, \beta_n$ are constants which fix the state of the gas. Carrying through the discussion as before, we have in place of (63)

$$\begin{aligned} \frac{\nu_1 \rho_1}{\sqrt{h_1}} &= \psi_1(M) \\ \frac{\nu_1}{h_1}\left(\rho_1^2 + \frac{1}{2}\right) &= \psi_2(M) \\ \frac{\nu_1 \rho_1}{h_1^{3/2}}\left(\rho_1^2 + \frac{5}{2}\right) &= \psi_3(M), \end{aligned} \tag{83}$$

where M is the rate of evaporation, and ψ_1, ψ_2, ψ_3 follow from the transfer equations for (82); whence again proceeding as above, we have

$$\frac{\rho_1^2\left(\rho_1^2 + \frac{5}{2}\right)}{\left(\rho_1^2 + \frac{1}{2}\right)^2} = \psi(M) \tag{84}$$

in place of (65). We thus finally obtain an expression identical with (68) except that we have ψ in place of α; and in place of (78) we have

$$\psi(M) \leqq \frac{25}{16},$$

an expression limiting the range of M. The curve Fig. 7 now follows, where α is replaced by $\psi(M)$; and we have a vertical tangent at the critical point, at which the isotropic gas is moving with its sound velocity. If, now, we assume from (77) and the physical nature of the problem that ρ_1 must be a one valued function of M; then it follows that one of the two portions of the curve meeting at the critical point cannot exist physically. Since the value $\rho_1 = 0$, corresponding to the stationary saturated vapor, is possible, it follows that the upper portion

of the curve is not; hence we again obtain the portion of the curve lying between the origin and the critical point. We have thus obtained the above results with the arbitrary distribution (82) instead of the assumed distribution (21). We thus see that the fact that we have obtained *exact* results by the use of the *assumed* distribution (21) cannot be taken as an indication that (21) is exact.

9. **Calculations, Tables, and Curves for the Gas in its Final Isotropic State.** Using the tables of Section 5 we now apply the results of Section 7; and thus obtain tables of values of the various constants of the isotropic gas in terms of ρ. The comments made in Section 5 concerning the computations also apply here. The results are given in Table II and in Figs. 8 and 9. The curves Fig. 9 are obtained by plotting $\frac{T_1}{T_0}$, $\frac{p_1}{p_0}$, and $\frac{\nu_1}{\nu_0}$ against $\frac{M}{M_0^+}$ for the same values of ρ, thus eliminating ρ. All figures tabulated are significant.[7]

In Section 8 we saw that these gas constants are two valued functions of ρ in the region corresponding to $1 < \alpha < \frac{25}{16}$; however, referring to Table I we see that the only tabulated values of ρ included in this region are .40, .45, and .50. In Table II values corresponding to the region on the stable side of the critical point are tabulated above the double horizontal line, whereas the three sets of values corresponding to the unstable region beyond the critical point are tabulated below this double line. It is evident that only the stable region can be obtained physically; hence to avoid confusion the curves Figs. 8 and 9 include only the stable region, and thus *stop*[11] at the critical point, at which point they have vertical tangents.

The value of ρ at the critical point is .508047 $\cdots$, and the data corresponding to this point are printed in heavy type in Table I and Table II. In carrying out the computation of this critical data the required values of $\frac{2}{\sqrt{\pi}}\int_0^{\rho} \epsilon^{-x^2}\,dx$ and $\frac{2}{\sqrt{\pi}}\epsilon^{-\rho^2}$ were obtained from the tables of Burgess[8] by means of Lagrangean interpolation,[12] the expression

[11] In the case of a liquid evaporating through a straight tube and discharging into a region of pressure P, the pressures P and p_1 coincide up to the critical point, beyond which p_1 and the rate of discharge remain constant as P is lowered.

[12] Rutledge and Crout, "Tables and Methods of Extending Tables for Interpolation Without Differences", *Journal of Mathematics and Physics*, vol. 9, p. 166.

Values of the derivatives required in the expression limiting the error were taken from Czuber, "Wahrscheinlichkeitsrechnung", vol. 1, p. 462.

The critical value of ρ was found to lie between .5080476 and .5080479.

for the error being used to hold the error below specified limits. Here again all figures tabulated are significant.[7]

Comparing Figs. 1 and 8, we see that the temperature of the isotropic gas lies between the two temperatures of the anisotropic gas; also we see from Figs. 4 and 8, or from Figs. 5 and 9, that the pressure of the isotropic gas is smaller than the pressure on the liquid surface.

If we consider the pressure-discharge curve of $\frac{p_1}{p_0}$ against $\frac{M}{M_0^+}$, including the unstable region, it is evident that in the joining of the stable and unstable branches of the curve at a critical point having a vertical tangent, at which point the gas is moving with its sound velocity, we have a close analogy to the case of a gas discharging through an orifice from a region of high pressure to one of lower pressure.[10]

If desired, the molecular energy flowing out per unit area of liquid surface may readily be obtained, this being given by the third equation of (60). Since $m\nu_1 u_1 = M$, $\frac{1}{2h_1} = kT_1$, and $p_1 = \nu_1 kT_1$, this may be written

$$E^+ - E^- = u_1\nu_1\left(\frac{1}{2}mu_1^2 + \frac{5}{2}kT_1\right),$$

or

$$E^+ - E^- = \frac{M}{m\nu_1}\left(\frac{M^2}{2m\nu_1} + \frac{5}{2}p_1\right), \tag{85}$$

no subscript being required since the energy flow is independent[8] of x. The heat which must be supplied to the liquid to maintain the evaporation is obtained by adding to (85) the energy required to get the molecules through the liquid surface.

It is to be emphasized that the above analysis applies directly and without modification to the process of sublimation; thus if throughout this paper we replace the word "evaporation" by "sublimation", and the word "liquid" by "solid", all methods and results apply in full to the sublimation of monatomic gases.

In the tables and curves which follow, the denominators of the tabulated ratios are the constants of the stationary saturated vapor. Since the various physical tables usually give only the saturation vapor pressure as a function of temperature, it will be convenient if we give the various constants in terms of pressure and *absolute* temperature. Thus from (10), (41), the gas equation

$$p_0 = \nu_0 kT_0, \tag{86}$$

and the relations

$$Am = \bar{m}, \qquad (87)$$
$$Ak = R,$$

where $\bar{m}$ is the molecular weight, A is Avogadro's number, and R is the gas constant $8.314 \cdot 10.^7$ ergs/degree per mole; it follows that

$$\text{Density} = m\nu_0 = \frac{\bar{m}p_0}{RT_0}$$
$$\text{Rate of emission} = M_0^+ = p_0\sqrt{\frac{\bar{m}}{2\pi RT_0}}. \qquad (88)$$

We have seen that the distance through which the gas must travel from the liquid surface before it becomes isotropic is of the order of several mean free paths. Since the state of the gas is changing continuously in this region, the mean free path is also; however we are interested primarily in the order of magnitude involved. Since the mean free path of an isotropic gas varies inversely with the density, we have

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0};$$

hence since $.325 < \frac{\nu_1}{\nu_0} \leqq 1$, we see that λ_1 and λ_0 are of the same order of magnitude. Similarly we should expect that λ, the mean free path of the anisotropic gas, should approximately vary inversely as the density; thus we *assume*

$$\frac{\lambda_0}{\lambda} \approx \frac{\nu}{\nu_0};$$

hence since in the stable region $0 \leqq \rho \leqq .508047 \cdots$ we have $.597 < \frac{\nu}{\nu_0} \leqq 1$, from which we see that λ and λ_0 are of the same order of magnitude. Since $.325 < \frac{\nu_1}{\nu_0} \leqq \frac{\nu}{\nu_0} \leqq 1$, it follows that $\lambda_0 \leqq \lambda \leqq \lambda_1 < 3.08\,\lambda_0$, from which we see that a distance equal to several, say five or ten, times λ_1 or times $3\,\lambda_0$ should be sufficient to allow the gas to become isotropic. Conversely, if the apparatus is such that there are barriers, walls, or condensing surfaces within such a distance of the liquid surface; then it is necessary to use caution in applying the above results, making sure

that the conditions under which they were derived are met. It is evident that the mean free path as used here gives the same type of restriction upon the apparatus dimensions that is obtained in the case of formulas for viscosity, diffusion, and heat conduction in gases.[4] For convenience we give the following expression for λ_0:

$$\lambda_0 = \frac{1}{\pi \nu_0 S^2 \sqrt{2}} = \frac{kT_0}{\pi p_0 S^2 \sqrt{2}}, \tag{89}$$

where S is the diameter of the molecule.

It is evident that if many problems arise involving a particular gas, it will be convenient to plot curves of p_0, $m\nu_0$, M_0^+, and λ_0 once and for all.[13]

10. **Examples and Illustrations.** In order to indicate the types of problems which can be handled by means of the above tables and curves, and in order to illustrate the methods involved, let us consider several problems involving the evaporation of mercury. For convenience, data on the liquid and on the stationary saturated vapor are given in Table III, in which the first and fourth columns are taken from physical tables (International Critical Tables), and the remaining columns are computed, using (88) and (89) where necessary. In this computation a slide rule was used wherever convenient. p_0 is the vapor pressure in dynes/cm.^{2} Plots of p_0, $m\nu_0$, and M_0^+ against $\frac{1}{T_0}$ are shown in Fig. 10, the values plotted being those of Table III.

In the problems which follow, our primary concern is with the methods used, great accuracy being unimportant. For this reason graphical methods are used when convenient, and a slide rule is used wherever possible; also for the same reason no attempt is made to investigate the number of significant figures of the final answers. If greater accuracy is required, it may obviously be obtained by using tables having smaller subdivisions than Table III,[14] by using large scale graphs, computing machines, fine interpolation methods, and so on; furthermore if such is desired, there is no difficulty involved in determining the accuracy of the results.

[13] If such plots be made on semilog paper with $1/T_0$ as abscissas, the resulting curves will at least in some cases (Hg for example) be very close to straight lines, Fig. 10.

[14] Tables are available which give the vapor pressure of mercury for every degree from −39.°C. to +400.°C.

"International Critical Tables", vol. 3, p. 206.

The set of curves used to make the cuts of Figs. 1–10 were not used in making the following calculations, another set of the same curves being used instead; hence there may be slight discrepancies between the values so obtained and those obtained by the direct use of Figs. 1–10.

Problem 1. Mercury having a surface temperature of 180.°C. evaporates and discharges through a short straight tube into a vacuum. What is the state of the gas in the tube; also what is the pressure acting on the surface of the mercury, and the rate of evaporation per unit area?

From Table III we have

$$T_0 = 453.°\text{K.}, \quad p_0 = 11740.\ \text{dynes/cm.}^2, \quad m\nu_0 = 6.24 \cdot 10.^{-6}\ \text{gm./cm.}^3,$$
$$M_0^+ = .342\ \text{gm./sec. cm.}^2;$$

also since the discharge is into a vacuum, we know that operation is at the critical point; hence

$$\rho = .508, \quad \frac{T_1}{T_0} = .633, \quad \frac{m\nu_1}{m\nu_0} = .326, \quad \frac{p_1}{p_0} = .206, \quad \frac{p_s}{p_0} = .550,$$
$$\frac{M}{M_0^+} = .838.$$

The state of the gas in the straight tube is thus

$$\text{Pressure} = p_1 = .206 \cdot 11740. = 2420.\ \text{dynes/cm.}^2,$$
$$\text{Density} = m\nu_1 = .326 \cdot 6.24 \cdot 10.^{-6} = 2.03 \cdot 10.^{-6}\ \text{gm./cm.}^3,$$
$$\text{Temperature} = T_1 = .633 \cdot 453. = 287.°\text{K.};$$

also we have

$$\text{Pressure on Hg surface} = p_s = .550 \cdot 11740. = 6460.\ \text{dynes/cm.}^2,$$
$$\text{Rate of evaporation} = M = .838 \cdot .342 = .287\ \text{gm./sec. cm.}^2$$

Problem 2. Same as Problem 1 except that discharge is into a region having a pressure of 5. mm., or 6670. dynes/cm.^{2}, instead of into a vacuum.

We have

$$\frac{p_1}{p_0} = \frac{5.}{8.796} = .568;$$

hence $\rho = .257$, from which it follows that

$$\frac{T_1}{T_0} = .870, \quad \frac{m\nu_1}{m\nu_0} = .656, \quad \frac{p_s}{p_0} = .657, \quad \frac{M}{M_0^+} = .596.$$

We thus have finally for the gas in the straight tube

Pressure = $p_1 = 6670.$ dynes/cm.^{2},

Density = $m\nu_1 = .656 \cdot 6.24 \cdot 10.^{-6} = 4.09 \cdot 10.^{-6}$ gm./cm.^{3},

Temperature = $T_1 = .870 \cdot 453. = 394.°\text{K.}$;

also we have

Pressure on Hg surface = $p_s = .657 \cdot 11740. = 7720.$ dynes/cm.^{2},

Rate of evaporation = $M = .596 \cdot .342 = .204$ gm./sec. cm.^{2}

Problem 3. Mercury evaporates at the rate of 2.30 gm./sec., discharging through a short cylindrical tube of 2. cm. diameter into a space maintained at a pressure of 3. mm., or 4010. dynes/cm.^{2} What is the state of the gas in the tube, the temperature of the mercury surface, and the pressure on this surface?

Let us temporarily assume that the rate of evaporation is so great for the given discharge pressure that operation is at the critical point, in which case

$$\rho = .508\,, \quad \frac{T_1}{T_0} = .633\,, \quad \frac{m\nu_1}{m\nu_0} = .326\,, \quad \frac{p_1}{p_0} = .206\,, \quad \frac{p_s}{p_0} = .550\,,$$

$$\frac{M}{M_0^+} = .838\,.$$

But

$$M = \frac{2.30}{\pi} = .733 \text{ gm./sec. cm.}^2\,;$$

hence

$$M_0^+ = \frac{.733}{.838} = .875 \text{ gm./sec. cm.}^2$$

Referring to Fig. 10, we see that $\frac{1}{T_0} = .002073$; whence

$T_0 = 483.°\text{K.}$, $p_0 = 31100.$ dynes/cm.^{2}, $m\nu_0 = 1.570 \cdot 10.^{-4}$ gm./cm.^{3},

$M_0^+ = .875$ gm./sec. cm.^{2}.

The state of the gas within the tube is thus

$$\text{Pressure} = p_1 = .206 \cdot 31100. = 6410. \text{ dynes/cm.}^2,$$

$$\text{Density} = m\nu_1 = .326 \cdot 1.570 \cdot 10.^{-4} = 5.12 \cdot 10.^{-5} \text{ gm./cm.}^3,$$

$$\text{Temperature} = T_1 = .633 \cdot 483. = 306.^\circ\text{K.};$$

also we have

$$\text{Pressure on liquid surface} = p_s = .550 \cdot 31100. = 17110. \text{ dynes/cm.}^2,$$

$$\text{Temperature of liquid surface} = T_0 = 483.^\circ\text{K}.$$

Since $p_1 > 4010.$ dynes/cm.^{2}, we see that the assumption that operation is at the critical point is justified.

Problem 4. Same as Problem 3 except that the gas discharges into a space having a pressure of 10. mm., or 13350. dynes/cm.^{2}, instead of 3. mm.

In this case the assumption that operation is at the critical point leads to a contradiction, for such an assumption gives the results of Problem 3, in which case the pressure of the gas in the tube would be smaller than the discharge pressure, which is impossible. It thus follows that $p_1 = 13350.$ dynes/cm.^{2}; also since $M = .733$ gm./sec. cm.^{2}, we may for assumed values of T_0 determine values of $\frac{p_1}{p_0}$ and $\frac{M}{M_0^+}$, which we may plot. The intersection of the curve so obtained with the curve Fig. 9 relating $\frac{p_1}{p_0}$ and $\frac{M}{M_0^+}$ gives the required values of these two quantities; thus we have $\frac{p_1}{p_0} = .392$, $\frac{M}{M_0^+} = .767$, from which it follows immediately that

$$M_0^+ = \frac{.733}{.767} = .955 \text{ gm./sec. cm.}^2, \quad p_0 = \frac{13350.}{.392} = 34000. \text{ dynes/cm.}^2,$$

$$\rho = .406, \qquad \frac{1}{T_0} = .002061/^\circ\text{K.}, \qquad T_0 = 485.^\circ\text{K.},$$

$$m\nu_0 = 1.68 \cdot 10.^{-4} \text{ gm./cm.}^3, \qquad \frac{T_1}{T_0} = .786, \qquad \frac{m\nu_1}{m\nu_0} = .499,$$

$$\frac{p_s}{p_0} = .581.$$

The state of the gas in the tube is thus

Pressure = p_1 = 13350. dynes/cm.^{2} ,

Density = $m\nu_1$ = .499·1.68·10.$^{-4}$ = 8.38·10.$^{-5}$ gm./cm.^{3} ,

Temperature = T_1 = .786·485. = 381.°K. ;

also we have

Liquid surface temperature = T_0 = 485.°K. ,

Pressure on liquid surface = p_s = .581·34000. = 19800. dynes/cm.^{2}

The above four problems had to do with evaporation and discharge through a straight tube; however it is evident that we are in a position to solve the problems which arise when a gas after evaporation is forced to go through a system of tubes, orifices, and so on. As illustrations we consider the following two problems.

Problem 5. Mercury having a surface temperature of 180.°C. evaporates and discharges through a short straight tube into a vacuum. The tube has a restriction at the discharge end giving a straight edged orifice of area equal to one half that of the liquid surface. What is the state of the gas in the tube, the pressure on the liquid surface, and the rate of discharge?

Since the discharge is into a vacuum, the *orifice*, which is supplied with gas in the state p_1, ν_1, T_1, is operating at its critical point; hence the rate of discharge per unit area of orifice, which because of the two to one restriction in area is twice that of the liquid surface, is for mercury[15]

$$2\,M = .001129\,\frac{p_1}{\sqrt{T_1}}\,. \tag{90}$$

From Table III we have

T_0 = 453.°K. , p_0 = 11740. dynes/cm.^{2} , $m\nu_0 = 6.24\cdot 10.^{-5}$ gm./cm.^{3} ,

M_0^+ = .342 gm./sec. cm.^{2}

[15] The maximum discharge of a monatomic gas from a region of pressure p_1 dynes/cm.^{2} and absolute temperature T_1 through a straight-edged orifice with rounded entry is .0000796 $p_1\sqrt{\frac{\bar{m}}{T_1}}$ grams per second per square centimeter of orifice area. This occurs when the jet is operating at its critical point, which in turn occurs when the discharge is into a region of pressure equal to or smaller than .487 p_1. If the discharge is into a region of higher pressure, the expression for the mass discharge is more complicated, but is well known,[16] and may be used in connection with our graphical methods with no particular difficulty.

Dividing (90) by $2\,M_0^+$, we have

$$\frac{M}{M_0^+} = \frac{.001129\; p_0 \left(\frac{p_1}{p_0}\right)}{2\,M_0^+ \sqrt{T_0}\sqrt{\frac{T_1}{T_0}}}$$

$$\frac{M}{M_0^+} = .909\; \frac{\left(\frac{p_1}{p_0}\right)}{\sqrt{\frac{T_1}{T_0}}}\,. \tag{91}$$

We now assume values of ρ, and determine values of the right side of (91); and thus obtain a curve, the intersection of which with that of Fig. 3 gives the required values of ρ and $\frac{M}{M_0^+}$; thus we have

$$\rho = .241\,, \qquad \frac{M}{M_0^+} = .574\,,$$

from which it follows immediately that

$$\frac{p_1}{p_0} = .590\,, \qquad \frac{\nu_1}{\nu_0} = .673\,, \qquad \frac{T_1}{T_0} = .878\,, \qquad \frac{p_s}{p_0} = .668\,.$$

The state of the gas in the tube is thus

Pressure $= p_1 = .590 \cdot 11740. = 6930.$ dynes/cm.^{2},

Density $= m\nu_1 = .673 \cdot 6.24 \cdot 10.^{-5} = 4.20 \cdot 10.^{-5}$ gm./cm.^{3},

Temperature $= T_1 = .878 \cdot 453. = 398.°$K. ;

also we have

Pressure on liquid surface $= p_s = .668 \cdot 11740. = 7850.$ dynes/cm.^{2},

Rate of discharge $= M = .574 \cdot .342 = .1963$ gm./sec. per cm.^{2} of liquid area, which is equivalent to .393 gm./sec. per cm.^{2} of orifice area.

These results compared with those of Problem 1 show the effect of the two to one restriction in area. Since the orifice is operating at its critical point, it is clear that the results of this problem would be unchanged if the discharge were into a region of any pressure smaller than the critical pressure of the jet, which in this case is $.487 \cdot 6930. = 3370.$

dynes/cm.², instead of into a vacuum. For higher discharge pressures the relation (90) must be modified,[15] the method of solution being otherwise the same.

Problem 6. Mercury evaporates and discharges through a short cylindrical tube of 3. cm. diameter into a vacuum. The discharge end of this tube is narrowed to form a straight sided circular orifice of 2. cm. diameter; and the rate of discharge is 2.00 gm./sec. What is the state of the gas in the tube, the temperature of the liquid surface, and the pressure on this surface?

We have at once

$$M = \frac{2.\cdot 4.}{9.\pi} = .283 \text{ gm./sec. cm.}^2 ;$$

also, as with (90),

$$\frac{9}{4} M = .001129 \frac{p_1}{\sqrt{T_1}},$$

or, since we know M,

$$564. = \frac{p_0}{\sqrt{T_0}} \frac{\left(\frac{p_1}{p_0}\right)}{\sqrt{\frac{T_1}{T_0}}}. \tag{92}$$

If, now, we assume a value of T_0 we immediately have p_0 and M_0^+; whence we obtain $\frac{M}{M_0^+}$ and thus ρ, $\frac{p_1}{p_0}$, and $\frac{T_1}{T_0}$. By assigning a set of values to T_0 we are thus enabled to plot a curve of the right hand side of (92) against T_0, the intersection of which with the ordinate 564. gives the required value of T_0. In this way we obtain the result $T_0 = 466.2°\text{K.}$, from which it follows that

$$\frac{1}{T_0} = .0021450/°\text{K.}, \qquad p_0 = 18400. \text{ dynes/cm.}^2 ,$$

$$m\nu_0 = 9.54 \cdot 10.^{-5} \text{ gm./cm.}^3 , \qquad M_0^+ = .528 \text{ gm./sec. cm.}^2 ;$$

whence

$$\frac{M}{M_0^+} = .536 , \quad \frac{p_1}{p_0} = .622 , \quad \frac{m\nu_1}{m\nu_0} = .701 , \quad \frac{T_1}{T_0} = .889 , \quad \frac{p_2}{p_0} = .689 .$$

The state of the gas in the tube is thus

$$\text{Pressure} = p_1 = .622 \cdot 18400. = 11460. \text{ dynes/cm.}^2 ,$$

$$\text{Density} = m\nu_1 = .701 \cdot 9.54 \cdot 10.^{-5} = 6.69 \cdot 10.^{-5} \text{ gm./cm.}^3 ,$$

$$\text{Temperature} = T_1 = .889 \cdot 466.2 = 414.^\circ\text{K.} ;$$

also we have

$$\text{Pressure on liquid surface} = p_s = .689 \cdot 18400. = 12680. \text{ dynes/cm.}^2 ,$$

$$\text{Temperature of liquid surface} = T_0 = 466.2^\circ\text{K} .$$

Since the orifice is operating at its critical point, it is clear that the results of this problem would be unchanged if the discharge were into a region of any pressure smaller than the critical pressure of the jet, which in this case is $.487 \cdot 11460. = 5580.$ dynes/cm.^{2}, instead of into a vacuum. For higher discharge pressures the relation (92) must be modified,[15] the method of solution being otherwise the same.

It may be noted that in solving the above problems it was not necessary to determine the state of the anisotropic gas at the liquid surface. If desired, however, this could easily be done; thus in the case of Problem 1 we have

$$\rho = .508 , \qquad \frac{T_L}{T_0} = .606 , \qquad \frac{T_T}{T_0} = 1.043 , \qquad \frac{m\nu}{m\nu_0} = .598 .$$

The state of the anisotropic gas at the liquid surface is thus

$$\text{Density} = m\nu = .598 \cdot 6.24 \cdot 10.^{-5} = 3.73 \cdot 10.^{-5} \text{ gm./cm.}^3 ,$$

$$\text{Longitudinal temperature} = T_L = .606 \cdot 453. = 275.^\circ\text{K.} ,$$

$$\text{Transverse temperature} = T_T = 1.043 \cdot 453. = 473.^\circ\text{K.}$$

Similarly it was not necessary to determine the drift velocities u_0 and u_1, though, if desired, these may easily be obtained; thus for the isotropic gas we have, noting (59), (26), and (87)

$$u_1 = \frac{\rho_1}{\sqrt{h_i m}} = \rho_1 \sqrt{\frac{2kT_1}{m}}$$
$$u_1 = \rho_1 \sqrt{\frac{2RT_1}{\bar{m}}} . \tag{93}$$

In a similar manner we have for the anisotropic gas at the liquid surface

$$u_0 = \rho \sqrt{\frac{2RT_L}{\bar{m}}}, \tag{94}$$

due to (39), (23), and (87). As an example we have in the case of Problem 1: $\rho_1 = .913$,

$$u_1 = .913 \sqrt{\frac{2.\cdot 8.314\cdot 10.^7\cdot 287.}{200.6}}$$

$$= 14100.\ \text{cm./sec.},$$

$$u_0 = .508 \sqrt{\frac{2.\cdot 8.314\cdot 10.^7\cdot 275.}{200.6}}$$

$$= 7680.\ \text{cm./sec.}$$

It may be noted that $m\nu u_0 = m\nu_1 u_1 = M$, as was to be expected; in fact this relation could have been used to determine u_0 and u_1 from the known values of M, $m\nu$, and $m\nu_1$.

We may also determine the energy flowing out from the liquid surface in the case of Problem 1; thus from (85) we have

$$E^+ - E^- = \frac{.287}{2.03\cdot 10.^{-6}}\left(\frac{.287^2}{2.\cdot 2.03\cdot 10.^{-6}} + \frac{5}{2}\cdot 2420.\right)$$

$$= 1.143\cdot 10.^8 \text{ ergs/sec. cm.}^2$$

$$= 2.73 \text{ calories/sec. cm.}^2$$

From physical tables (Smithsonian Physical Tables) we find that the latent heat of vaporisation of Hg at 358.°C. is 71. cal./gm. I assume that this figure refers to a gas which after evaporating is essentially at rest and at a temperature[16] of 358.°C. To find the heat required to get

[16] We can easily see that for a given liquid surface temperature, and a given amount of energy, L_S, required per gram[17] to get the molecules through the liquid surface, *still the latent heat is not specified, for we must add* L_M, *the kinetic energy required per gram of gas evaporated, which depends upon* ρ_1, or the manner of evaporation. Dividing the third equation of (60) by the first, we find that the kinetic energy per gram of gas evaporated is

$$L_M = \frac{1}{2h_1 m}\left(\rho_1^2 + \frac{5}{2}\right)$$

$$= \frac{RT_0}{\bar{m}}\left(\frac{T_1}{T_0}\right)\left(\rho_1^2 + \frac{5}{2}\right).$$

The quantity $\frac{RT_0}{\bar{m}}$ depends only upon the kind of gas, and the temperature of the

1 gm. of Hg vapor through the liquid surface we must subtract from the 71. cal./gm. the energy which remains as kinetic energy of the molecules. Since 358.°C. = 631.°K. we thus must subtract

$$\frac{5}{2}\cdot 631.\ \frac{kA}{\bar{m}}\ \text{ergs./gm.}$$

$$= \frac{5}{2}\cdot 631.\cdot 2.39\cdot 10.^{-8}\ \frac{R}{\bar{m}}\ \text{cal./gm.}$$

$$= \frac{5.\cdot 631.\cdot 2.39\cdot 10.^{-8}\cdot 8.314\cdot 10.^{7}}{2.\cdot 200.6}$$

$$= 15.6\ \text{cal./gm.}$$

The energy required to get 1 gm. of Hg through the surface is thus 55.4 cal.[17] Since the rate of evaporation is .287 gm./sec. cm.^{2} we thus require $.287\cdot 55.4 = 15.9$ cal./sec. cm.^{2} merely to get the molecules through the liquid surface. This is almost six times the energy which must be supplied as kinetic energy of the molecules, which we saw was 2.73 cal./sec.cm.^{2}. The total energy which must be supplied to the liquid to maintain the evaporation is thus $2.73 + 15.9 = 18.6$ cal./sec. cm.^{2}.

In solving the above problems no use was made of the mean free path. We have seen that several mean free paths are required for the gas to reach its isotropic state; thus this quantity is of use here only to indicate when the dimensions of the tubes or other apparatus are so small that the evaporating gas is interferred with before it has a chance to become isotropic. Thus in the case of Problem 1 we see that

$$\lambda_0 = .001371\ \text{cm.}, \qquad \lambda_1 = \frac{.001371}{.326} = .00421\ \text{cm.},$$

from which we assume that a distance of the order of .01 cm. is required for the gas to become isotropic.

---

liquid surface; however, as ρ varies from 0 to the critical value .508047 $\cdots$, the quantity $\frac{T_1}{T_0}\left(\rho_1^2 + \frac{5}{2}\right)$ varies from 2.500 to 2.109.

[17] It is assumed that this quantity is essentially constant, hence independent of temperature. The purpose of mentioning the energy required to overcome the surface attraction is not to become involved with the mechanism of emission, but to show to how great an extent this energy constitutes the total energy required to maintain the evaporation.

List of Symbols

The following partial list of symbols applies to all parts of this paper except Section 1.

T_0 = absolute temperature of liquid surface, or of stationary saturated vapor, in degrees Kelvin.
p_0 = pressure of stationary saturated vapor in dynes/cm.^{2}
ν_0 = molecular density of stationary saturated vapor in molecules/cm.^{3}
λ_0 = mean free path of stationary saturated vapor in cm.
p_s = pressure on liquid surface in dynes/cm.^{2}
M_0^+ = rate of emission of molecules from liquid surface in gm./sec.cm.^{2}
M = rate of evaporation in gm./sec.cm.^{2}
T_L = longitudinal temperature of anisotropic gas at liquid surface in degrees absolute, or Kelvin.
T_T = transverse temperature of anisotropic gas at liquid surface in degrees absolute, or Kelvin.
ν = molecular density of anisotropic gas at liquid surface in molecules/cm.^{3}
u_0 = drift velocity of anisotropic gas at liquid surface in cm./sec.
T_1 = temperature of gas in its final isotropic state in degrees absolute, or Kelvin.
ν_1 = molecular density of gas in its final isotropic state in molecules/cm.^{3}
p_1 = pressure of gas in its final isotropic state in dynes/cm.^{2}
u_1 = drift velocity of gas in its final isotropic state in cm./sec.
λ_1 = mean free path of gas in its final isotropic state in cm.
ρ is used merely as a parameter in the adjoining tables and curves. See Section 8. ρ has no dimensions.
ρ_1 is discussed in Section 8. ρ_1 has no dimensions.
u_s = velocity of sound in gas in its final isotropic state in cm./sec.
m = mass of molecule in gm.
$\bar{m}$ = molecular weight.
k = Boltzman's constant = $1.371 \cdot 10.^{-16}$ ergs/degree.
R = gas constant = $8.314 \cdot 10.^7$ ergs/degree per mole.
A = Avogadro's number = $6.064 \cdot 10.^{23}$ molecules per mole.
ϵ = base of natural system of logarithms = 2.71828 · · · .
π = ratio of circumference to diameter of circle = 3.14159 · · · .

Throughout this paper all units are C. G. S., and all temperatures are in degrees absolute, or Kelvin, except where it is expressly stated otherwise.

TABLE I

ρ	T_L/T_0	T_T/T_0	v/v_0	M/M_0^*	p_L/p_0	h_L/h_0	σ	α
0	1.00000	1.00000	1.00000	0	1.00000	1.00000	1.00000	0
.03	.97132	1.00248	.96261	.10089	.93668	1.02953	1.03208	.00909
.06	.94338	1.00497	.92759	.19163	.88137	1.06002	1.06529	.03647
.10	.90727	1.00832	.88430	.29859	.81834	1.10221	1.11138	.10078
.15	.86391	1.01253	.83513	.41275	.75394	1.15753	1.17204	.22207
.20	.82248	1.01678	.79087	.50851	.70251	1.21584	1.23624	.38083
.25	.78291	1.02105	.75096	.58887	.66143	1.27728	1.30417	.56626
.30	.74516	1.02534	.71491	.65630	.62861	1.34200	1.37600	.76685
.35	.70915	1.02964	.68230	.71288	.60240	1.41013	1.45193	.97190
.40	.67484	1.03395	.65277	.76036	.58147	1.48184	1.53215	1.17251
.45	.64215	1.03827	.62597	.80019	.56477	1.55726	1.61686	1.36203
.50	.61104	1.04258	.60164	.83358	.55144	1.63655	1.70624	1.53611
***.508047...**	**.60618**	**1.04328**	**.59794**	**.83843**	**.54956**	**1.64969**	**1.72108**	**1.56250**
.55	.58144	1.04689	.57952	.86157	.54081	1.71986	1.80050	1.69239
.60	.55330	1.05117	.55938	.88500	.53235	1.80734	1.89983	1.83010
.70	.50115	1.05968	.52429	.92099	.52024	1.99540	2.11449	2.05204
.80	.45415	1.06806	.49504	.94609	.51259	2.20191	2.35177	2.21201
.90	.41186	1.07626	.47057	.96349	.50778	2.42799	2.61315	2.32373
1.00	.37387	1.08424	.45004	.97549	.50478	2.67471	2.90004	2.40002
1.20	.30922	1.09940	.41821	.98926	.50176	3.23391	3.55537	2.48521
1.40	.25732	1.11330	.39543	.99550	.50062	3.88617	4.32647	2.52189
1.60	.21569	1.12579	.37895	.99820	.50021	4.63639	5.21958	2.53703
1.80	.18222	1.13682	.36688	.99932	.50007	5.48777	6.23860	2.54301
2.00	.15523	1.14644	.35791	.99976	.50002	6.44208	7.38544	2.54527
∞	0	$\left(2-\frac{\pi}{4}\right)$	$1/\pi$	1	1/2	∞	∞	$8/\pi$

* This row contains data for the critical point, at which the isotropic gas moves with its sound velocity. See Sections 8 and 9.

TABLE II

ρ	T_1/T_0	v_1/v_0	p_1/p_0	ρ_1	h_1/h_0
0	1.00000	1.00000	1.00000	0	1.00000
.03	.98377	.95040	.93497	.030192	1.01649
.06	.96795	.90388	.87490	.060788	1.03312
.10	.94735	.84611	.80157	.10228	1.05557
.15	.92222	.77983	.71917	.15548	1.08434
.20	.89744	.71902	.64527	.21060	1.11429
.25	.87259	.66254	.57813	.26841	1.14601
.30	.84712	.60923	.51609	.33018	1.18048
.35	.82009	.55771	.45737	.39817	1.21938
.40	.78979	.50598	.39962	.47701	1.26616
.45	.75189	.44983	.33822	.57871	1.32998
.50	.68215	.36881	.25158	.77197	1.46595
*.508047...	.63270	.32573	.20609	.91287	1.58053
†.50	.57048	.28397	.16200	1.09636	1.75292
†.45	.40154	.21257	.08535	1.67583	2.49041
†.40	.21612	.16884	.03649	2.73271	4.62713

* This row contains data for the critical point, at which the isotropic gas moves with its sound velocity. See Sections 8 and 9.

† The portion of this table lying below the double horizontal line corresponds to the unstable region beyond the critical point, which cannot exist physically. See Sections 8 and 9.

TABLE III

Data for Mercury

T_0 in °C.	T_0	$1/T_0$	p_0 in mm. of Hg	p_0	mp_0	M_0^2	λ_0
140.	413.	.002421	1.845	2460.	$1.436 \times 10.^{-5}$	.0751	$5.96 \times 10.^{-4}$
145.	418.	.002392	2.282	3050.	1.755	.0924	4.88
150.	423.	.002364	2.807	3750.	2.13	.1129	4.01
155.	428.	.002336	3.438	4580.	2.58	.1374	3.32
160.	433.	.002309	4.189	5590.	3.11	.1667	2.75
165.	438.	.002283	5.080	6780.	3.73	.201	2.294
170.	443.	.002257	6.128	8180.	4.45	.241	1.927
175.	448.	.002232	7.356	9820.	5.27	.287	1.622
180.	453.	.002208	8.796	11740.	6.24	.342	1.371
185.	458.	.002183	10.472	13990.	7.35	.404	1.166
190.	463.	.002160	12.423	16590.	8.63	.478	$9.92 \times 10.^{-4}$
195.	468.	.002137	14.681	19590.	$1.008 \times 10.^{-4}$	.561	8.49
200.	473.	.002114	17.287	23100.	1.175	.657	7.28
205.	478.	.002092	20.285	27100.	1.365	.767	6.27
210.	483.	.002070	23.723	31700.	1.580	.893	5.42
215.	488.	.002049	27.654	36900.	1.821	1.037	4.70
220.	493.	.002028	32.133	42900.	2.095	1.198	4.08
225.	498.	.002008	37.222	49700.	2.40	1.380	3.56
230.	503.	.001988	42.989	57400.	2.75	1.584	3.11
235.	508.	.001969	49.509	66100.	3.13	1.818	2.73
240.	513.	.001949	56.855	75900.	3.56	2.074	2.40
245.	518.	.001931	65.113	86900.	4.04	2.37	2.12
250.	523.	.001912	74.375	99200.	4.57	2.69	1.873

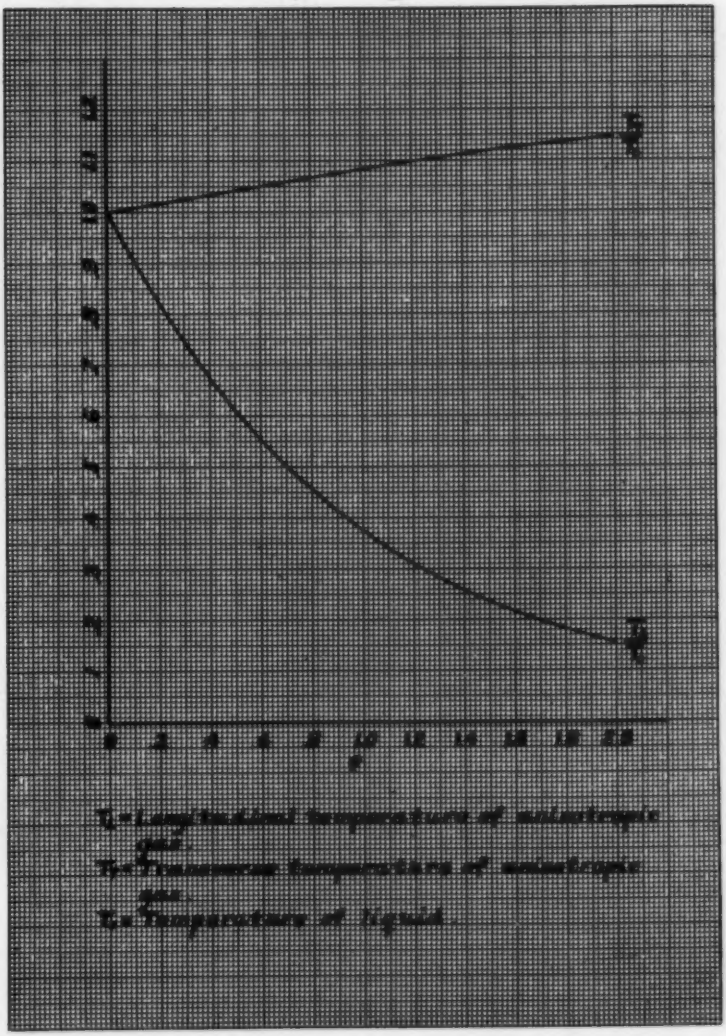

FIG. 1

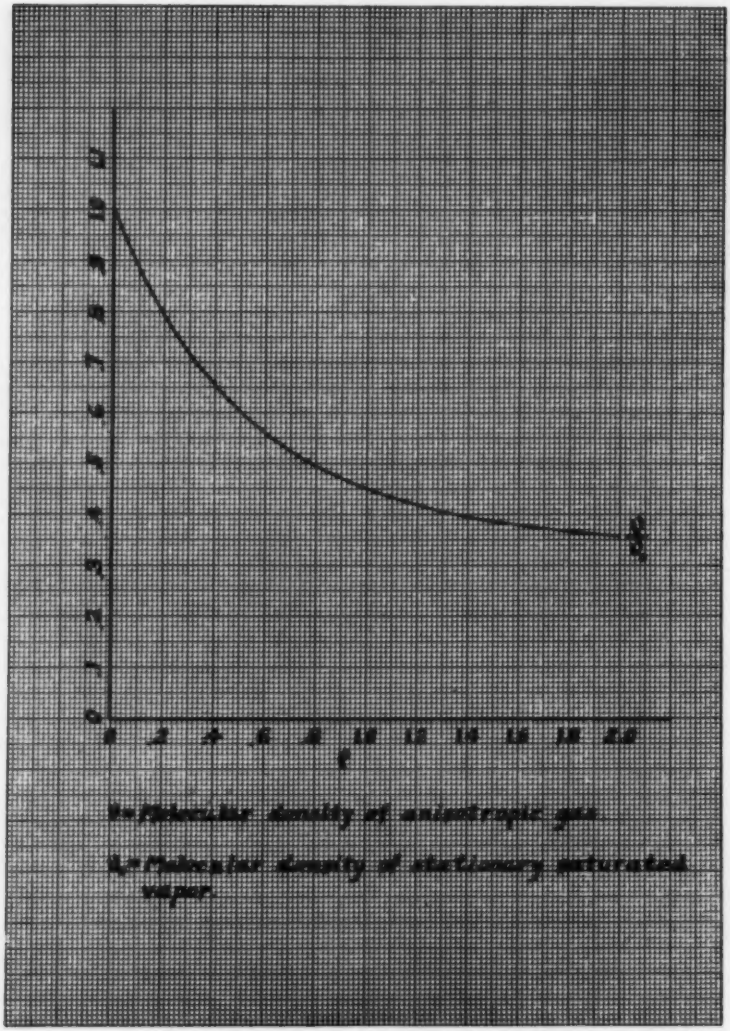
0 1 2 3 4 5 6 7 8 9 10
0 2 4 6 8 10 12 14 16 18 20
ρ
ρ_0
ρ_s
ρ_0=Molecular density of anisotropic gas.
ρ_s=Molecular density of stationary saturated vapor.

FIG. 2

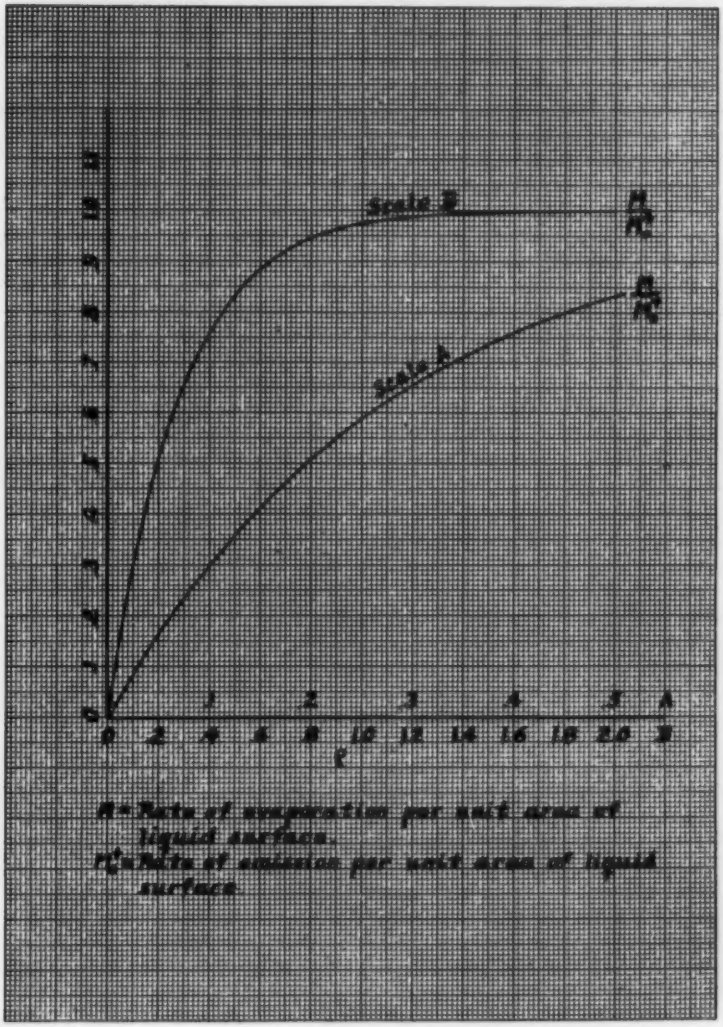
scale 2
scale 1
0 2 4 6 8 10 12 14 16 18 20 22
1 2 3 4 5
t'
A = Rate of evaporation per unit area of liquid surface.
r'A = Rate of emission per unit area of liquid surface.

FIG. 3

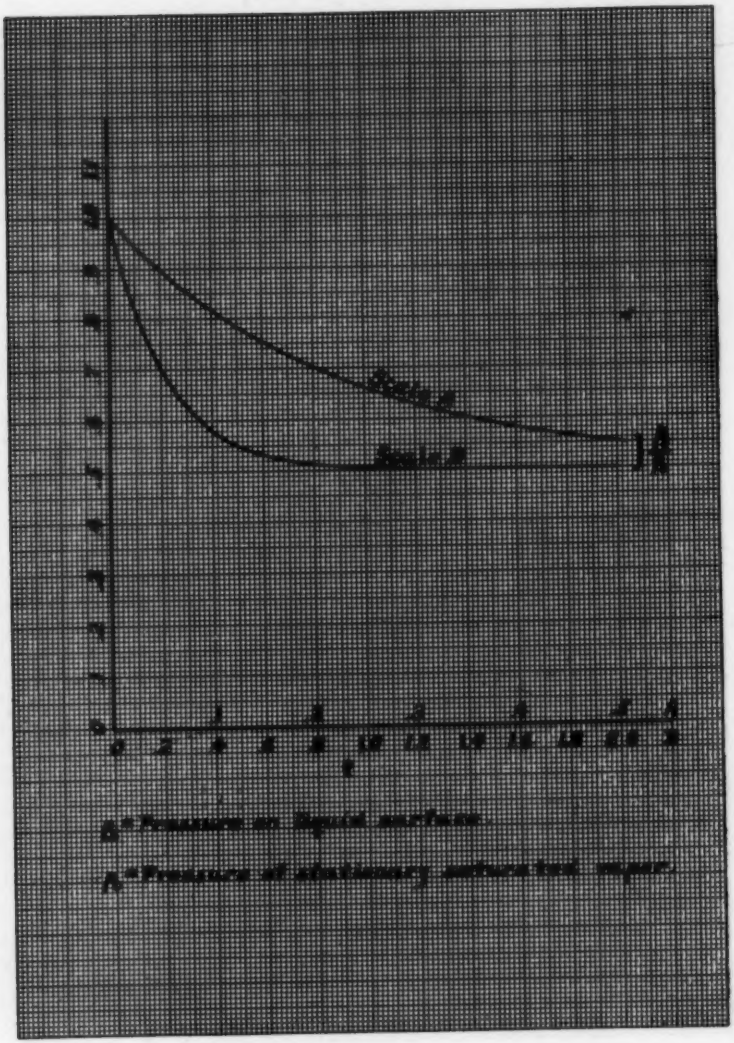

FIG. 4

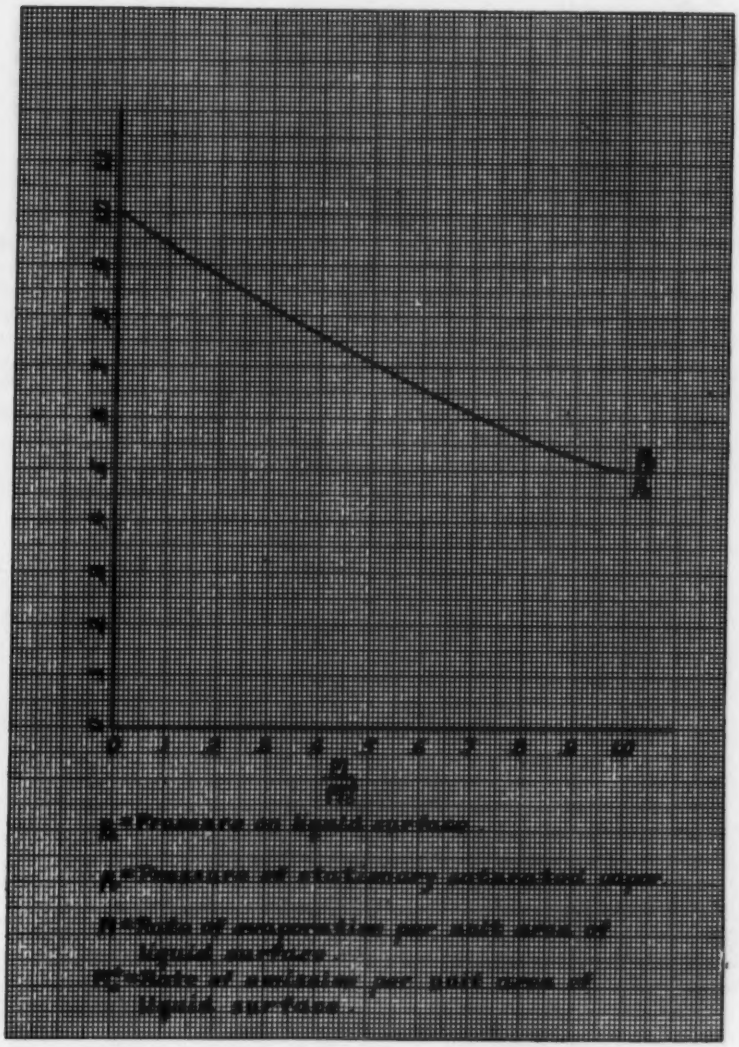

FIG. 5

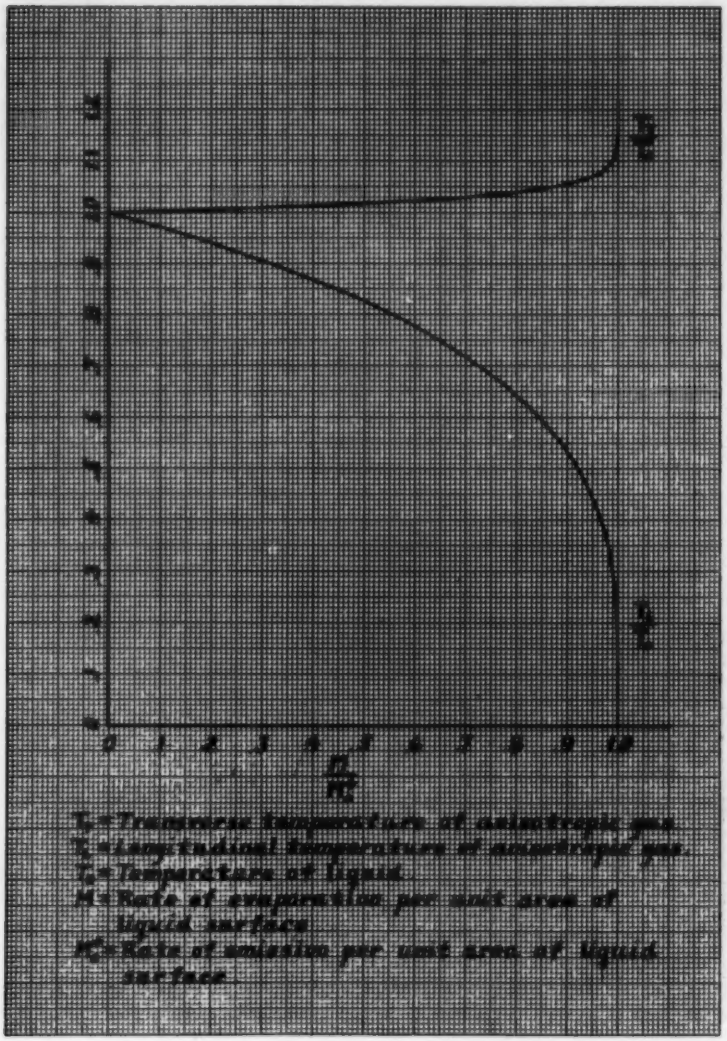
T_t
T_l
$\frac{T_t}{T_l}$
$\frac{T_l}{T_t}$
$\frac{T_t}{T_l}$
T_t = Transverse temperature of anisotropic gas.
T_l = Longitudinal temperature of monatomic gas.
T_c = Temperature of liquid.
M_1 = Rate of evaporation per unit area of liquid surface.
M_2 = Rate of emission per unit area of liquid surface.

FIG. 6

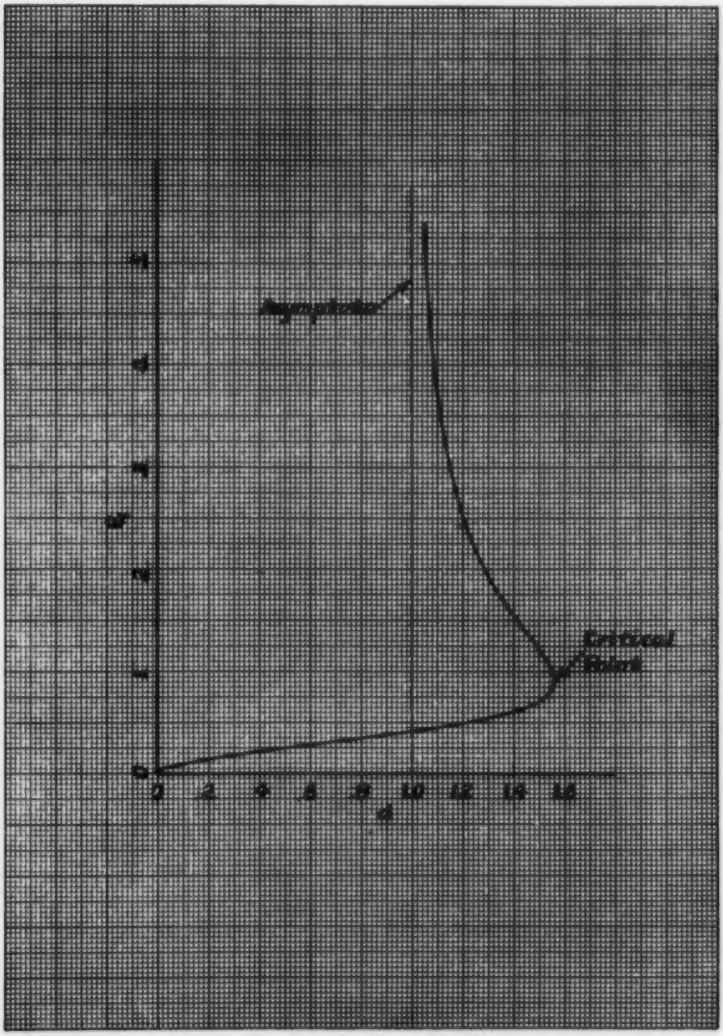

FIG. 7

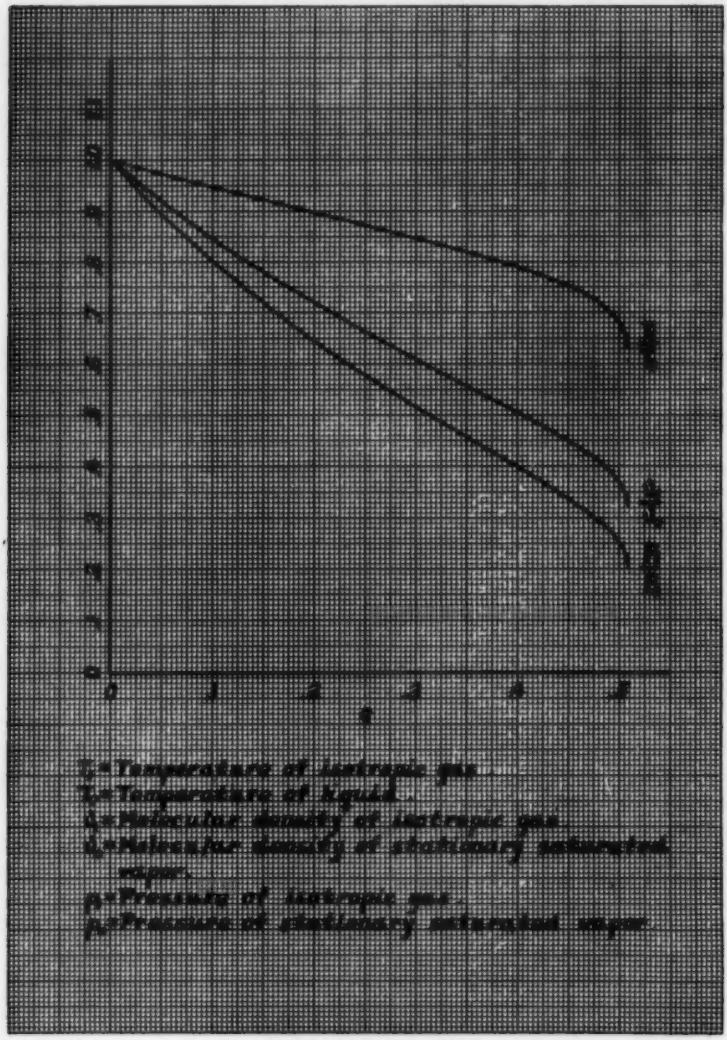
0
1
2
3
4
5
θ
T=Temperature of isotropic gas.
T_b=Temperature of liquid.
ρ=Molecular density of isotropic gas.
ρ_b=Molecular density of stationary saturated vapor.
p=Pressure of isotropic gas.
p_b=Pressure of stationary saturated vapor

FIG. 8

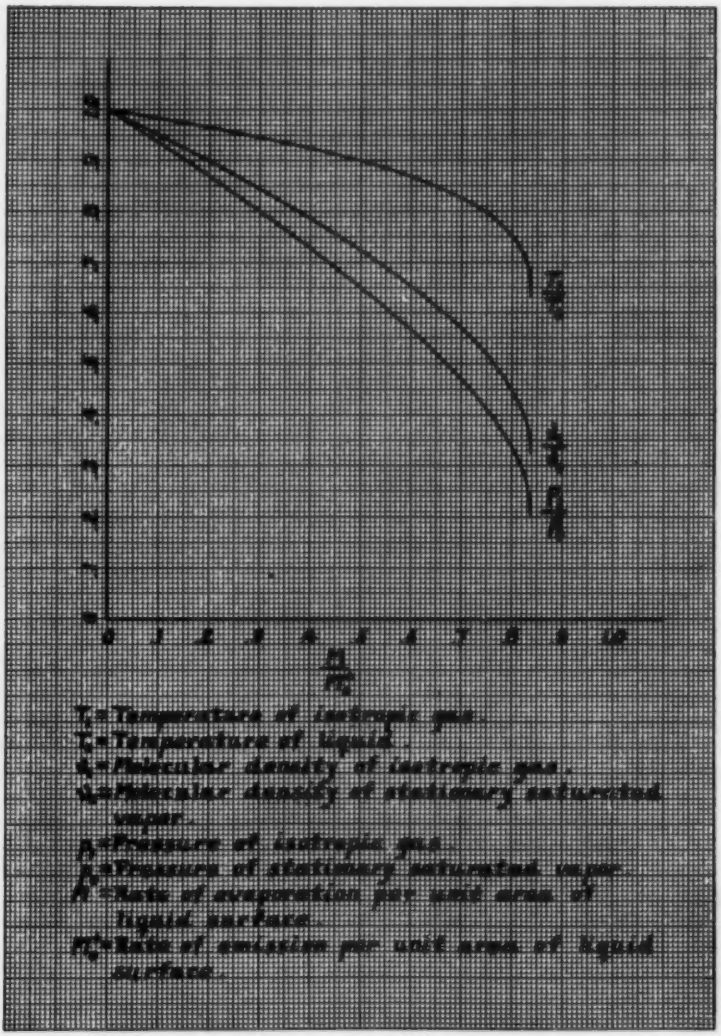
T_1: Temperature of isotropic gas.
T_2: Temperature of liquid.
k_1: Molecular density of isotropic gas.
k_2: Molecular density of stationary saturated vapor.
p_1: Pressure of isotropic gas.
p_2: Pressure of stationary saturated vapor.
R': Rate of evaporation per unit area of liquid surface.
R'': Rate of emission per unit area of liquid surface.

FIG. 9

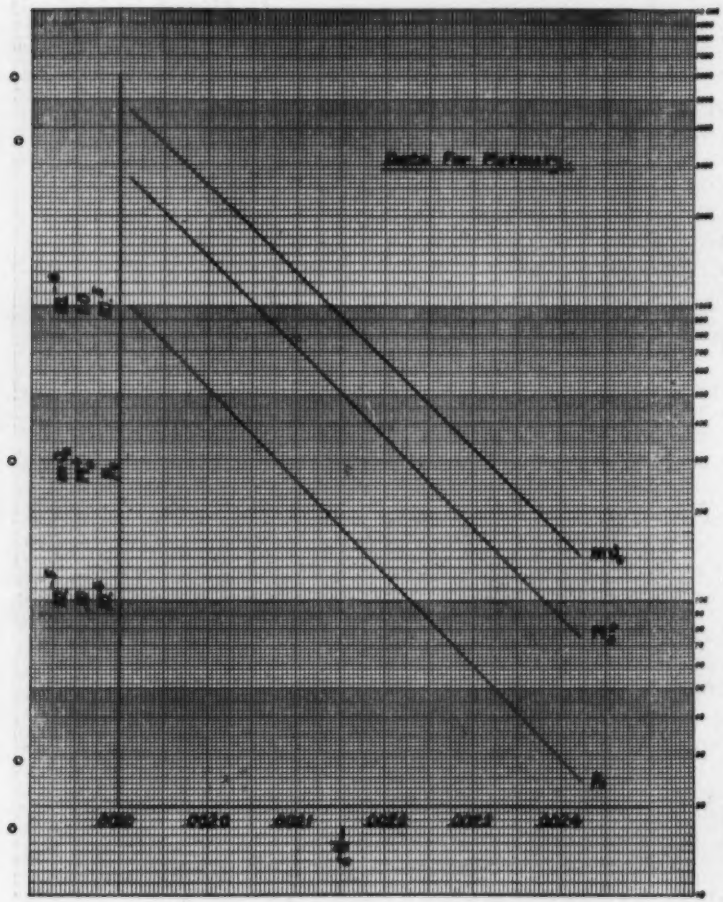

FIG. 10

SOME NEW RESULTS IN THE PROBLEM OF PLATEAU

By Jesse Douglas

1. During the last academic year,* the author completed the work of extending his solution of the problem of Plateau for one contour,[1] for two contours,[2] and for minimal surfaces of the topological type of a Möbius strip,[3] to the very general case where the minimal surface has any finite genus, is either two-sided or one-sided (orientable or non-orientable), and is bounded by any finite number of contours prescribed in form and position in euclidean space of any number n of dimensions.[4] Each contour is supposed to be a continuous closed curve, not intersecting itself or any of the other contours, and with a given sense of description.

The present communication describes the main results of this work; it is expected that details and proofs will be given in an extensive paper to be published elsewhere under the title: "Minimal surfaces of general topological structure with any finite number of assigned boundaries."

2. We shall consider surfaces, either two-sided or one-sided, of any finite connectivity, and with any finite number k of boundaries.

The definition of the *Betti number* r of such a surface is: the maximum number of circuits, no linear combination of which is equivalent to a bounding circuit.[5] This is the same as the excess of the connectivity over the number of boundaries: $c = r + k$. *Connectivity* c means that

* 1934–35, over which period the author was a research worker at the Institute for Advanced Study at Princeton, being on leave from the Massachusetts Institute of Technology.

[1] "Solution of the problem of Plateau," *Trans. Amer. Math. Soc.*, v. 33 (1931), pp. 263–321.

[2] "The problem of Plateau for two contours," *Jour. Math. Phys.*, v. 10 (1931), pp. 315–359.

[3] "One-sided minimal surfaces with a given boundary," *Trans. Amer. Math. Soc.*, v. 34 (1932), pp. 731–756.

[4] This general form of statement of the problem was first given by the author in a paper presented to the American Mathematical Society in October, 1929; abstract published in *Bull. Amer. Math. Soc.*, v. 36 (1930), p. 50.

[5] See S. Lefschetz, *Topology*, (New York, 1930), p. 35.

there exist $c - 1$ successive cross-cuts which leave the surface connected, whereas every succession of c cross-cuts disconnects. A *cross-cut* is an arc each of whose end points lies on a boundary or on a previously drawn cross-cut, while all the other points of the arc are interior points of the surface.

For a *two-sided surface*, the Betti number is always *even*:

$$r = 2h, \qquad c = 2h + k,$$

where the surface is topologically equivalent to a sphere with h handles and k perforations.

For a *one-sided surface*, the Betti number may be either *odd or even*. Examples: Möbius strip, $r = 1$; Klein surface, $r = 2$; Möbius strip with h handles, $r = 2h + 1$; Klein surface with h handles, $r = 2h + 2$. A Klein surface is the one-sided closed surface which results from a sphere in which two holes have been cut, when these holes are joined by means of a tube leading from the outside to the inside of the sphere —and necessarily intersecting the spherical surface.[6]

3. (Systematically, we shall use a bar to denote one-sided surfaces and quantities pertaining to them, while all symbols relating to two-sided surfaces will be without a bar.)

Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be the given contours, and let Σ_r denote any two-sided surface, with Betti number r, bounded by these contours. This will imply that, a definite one of the two orientations of Σ_r being chosen, the preassigned sense on $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ is such that Σ_r is on the left.

The areas $S(\Sigma_r)$ of all such surfaces form a definite set of positive numbers (including $+\infty$). We define:

$$(1) \qquad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = \textit{lower bound } S(\Sigma_r).$$

m_r is a functional of the given contours and the even integer $r \geqq 0$; its value is always a positive number, either finite or $+\infty$. We shall assume (except for a remark at the end of this note) that

$$(2) \qquad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < +\infty,$$

that is, *finite*. A sufficient condition for this to be the case is that the given contours be rectifiable.

By considering the possible modes of degeneration of Σ_r into two

[6] See Hilbert and Cohn-Vossen, *Anschauliche Geometrie*, (Berlin, 1932), pp. 271, 272.

surfaces, each bounded by a partial group of the given contours, or into a surface of lower topological form, we obtain the following inequalities governing m_r:

$$(3)\quad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq m_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + m_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

$$(4)\quad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq m_{r-2}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k).$$

The inequality (3) holds for every partition of the contours, and of the integer r into *even* components:

$$(5)\quad (\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = (\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + (\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

$$(6)\quad r = r' + r''.$$

Similarly, with reference to one-sided surfaces $\bar{\Sigma}_r$, with odd or even Betti number r, bounded by the given contours, we make the definition:

$$(7)\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = \textit{lower bound } S(\bar{\Sigma}_r).$$

Since a one-sided surface may break up either into two one-sided surfaces, or into a one-sided and a two-sided surface, we have the two inequalities:

$$(8)\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq \bar{m}_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + \bar{m}_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

$$(8')\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq \bar{m}_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + m_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

for all partitions (5), (6); where in (6), r' and r'' may now be *odd or even* (except that in (8'), r'', pertaining to a two-sided surface, must be even).

Again, a one-sided surface $\bar{\Sigma}_r$ may degenerate into a $\bar{\Sigma}_{r-1}$; and, if r is odd, also into a Σ_{r-1}; hence we have the inequalities

$$(9)\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq \bar{m}_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

$$(9')\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq m_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

the latter applying only if r is odd.

All these inequalities may be applied repeatedly, or combined with one another in any manner. The resulting inequalities correspond to a separation of the given contours into any number of partial groups, and to any possible reduction of the Betti numbers.

4. The method of analysis which we actually use, however, does not depend on the areas of surfaces, but rather on a certain functional $A(g; \mu)$, whose argument g is an arbitrary parametric representation of the ensemble of contours, while its argument μ is the system of

there exist $c - 1$ successive cross-cuts which leave the surface connected, whereas every succession of c cross-cuts disconnects. A *cross-cut* is an arc each of whose end points lies on a boundary or on a previously drawn cross-cut, while all the other points of the arc are interior points of the surface.

For a *two-sided surface*, the Betti number is always *even*:

$$r = 2h, \qquad c = 2h + k,$$

where the surface is topologically equivalent to a sphere with h handles and k perforations.

For a *one-sided surface*, the Betti number may be either *odd or even*. Examples: Möbius strip, $r = 1$; Klein surface, $r = 2$; Möbius strip with h handles, $r = 2h + 1$; Klein surface with h handles, $r = 2h + 2$. A Klein surface is the one-sided closed surface which results from a sphere in which two holes have been cut, when these holes are joined by means of a tube leading from the outside to the inside of the sphere —and necessarily intersecting the spherical surface.[6]

3. (Systematically, we shall use a bar to denote one-sided surfaces and quantities pertaining to them, while all symbols relating to two-sided surfaces will be without a bar.)

Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be the given contours, and let Σ_r denote any two-sided surface, with Betti number r, bounded by these contours. This will imply that, a definite one of the two orientations of Σ_r being chosen, the preassigned sense on $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ is such that Σ_r is on the left.

The areas $S(\Sigma_r)$ of all such surfaces form a definite set of positive numbers (including $+\infty$). We define:

$$(1) \qquad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = \textit{lower bound } S(\Sigma_r).$$

m_r is a functional of the given contours and the even integer $r \geqq 0$; its value is always a positive number, either finite or $+\infty$. We shall assume (except for a remark at the end of this note) that

$$(2) \qquad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < +\infty,$$

that is, *finite*. A sufficient condition for this to be the case is that the given contours be rectifiable.

By considering the possible modes of degeneration of Σ_r into two

[6] See Hilbert and Cohn-Vossen, *Anschauliche Geometrie*, (Berlin, 1932), pp. 271, 272.

surfaces, each bounded by a partial group of the given contours, or into a surface of lower topological form, we obtain the following inequalities governing m_r:

$$(3)\quad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq m_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + m_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

$$(4)\quad m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq m_{r-2}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k).$$

The inequality (3) holds for every partition of the contours, and of the integer r into *even* components:

$$(5)\quad (\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = (\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + (\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

$$(6)\quad r = r' + r''.$$

Similarly, with reference to one-sided surfaces $\bar{\Sigma}_r$, with odd or even Betti number r, bounded by the given contours, we make the definition:

$$(7)\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = \textit{lower bound } S(\bar{\Sigma}_r).$$

Since a one-sided surface may break up either into two one-sided surfaces, or into a one-sided and a two-sided surface, we have the two inequalities:

$$(8)\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq \bar{m}_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + \bar{m}_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

$$(8')\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq \bar{m}_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + m_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

for all partitions (5), (6); where in (6), r' and r'' may now be *odd or even* (except that in (8'), r'', pertaining to a two-sided surface, must be even).

Again, a one-sided surface $\bar{\Sigma}_r$ may degenerate into a $\bar{\Sigma}_{r-1}$; and, if r is odd, also into a Σ_{r-1}; hence we have the inequalities

$$(9)\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq \bar{m}_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

$$(9')\quad \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) \leqq m_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

the latter applying only if r is odd.

All these inequalities may be applied repeatedly, or combined with one another in any manner. The resulting inequalities correspond to a separation of the given contours into any number of partial groups, and to any possible reduction of the Betti numbers.

4. The method of analysis which we actually use, however, does not depend on the areas of surfaces, but rather on a certain functional $A(g; \mu)$, whose argument g is an arbitrary parametric representation of the ensemble of contours, while its argument μ is the system of

moduli of a symmetric Riemann surface R, having a topological structure determined by that prescribed for the required minimal surface M_r or $\bar{M}_r$. $A(g;\mu)$ is a generalization of the original functional $A(g)$, by means of which the author obtained a complete solution of the Plateau problem for a single contour, and of the functionals $A(g_1, g_2; q)$ and $A(g; q)$, used respectively in the case of two contours and of a Möbius strip.

The defining formula of $A(g;\mu)$ is

$$(10)\quad A(g;\mu) = \frac{1}{4\pi}\int_C\int_C \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2 \frac{\partial^2}{\partial z\,\partial\zeta} \log\theta(v(z,\zeta) + \rho;\tau)\,dz\,d\zeta,$$

and its value is always a positive real number.

In explaining the formula (10), we consider first the case of a two-sided minimal surface M_r.

Let R denote any symmetric[7] Riemann surface of genus

$$(11)\qquad p = r + k - 1,$$

and with k *curves of transition*[8] $C_1, C_2, \cdots, C_k$. These are required to separate R into two semi Riemann surfaces, of which we choose a definite one, R'. The semi Riemann surface R' will have the same topological structure as that prescribed for M_r: Betti number r and k boundaries. We imagine these boundaries $C_1, C_2, \cdots, C_k$ sensed so that R', after being definitely oriented, is on the left.

An arbitrary parametric representation of $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$, as defined by equations

$$(12)\qquad x_i = g_{\alpha i}(z) \qquad (i = 1, 2, \cdots, n)$$

—where α, distinguishing the particular contour Γ_α, runs from 1 to k—amounts to a topological correspondence, preserving sense, between each contour Γ_α and the corresponding C_α. The equations (12) may be written in the condensed form

$$(13)\qquad x_i = g_i(z),$$

which may be regarded as establishing a sense-preserving topological correspondence between the ensemble of contours,

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots \Gamma_k,$$

[7] See F. Klein, *On Riemann's theory of algebraic functions and their integrals*, (English translation, Cambridge, 1893), §20, p. 66.

[8] F. Klein, loc. cit., §21, p. 67.

locus of the point (x_i), and the ensemble of curves of transition,

$$C = C_1 + C_2 + \cdots + C_k ,$$

locus of the point z.

The moduli μ of R are real, and in number $3p - 3 = 3(r + k - 2)$, except in the cases $r = 0$, $k = 1$ and $r = 0$, $k = 2$, when the number of moduli is 0 and 1, respectively.

The symbol $v(z, \zeta)$ in (10) denotes the system of p normal Abelian integrals of the first kind on R (whose periods with respect to the first system of circuits are $\pi\sqrt{-1}\,\delta_{jk}$, where $\delta_{jk} = 0$, if $j \neq k$; $= 1$, if $j = k$). Each integral v_j is taken from the lower limit ζ to the upper limit z.

The symbol τ denotes the p-by-p matrix τ_{jk} of periods of $v(z, \zeta)$ with respect to the second system of circuits.

θ denotes the multiple θ-series of Riemann:

$$\theta(u; \tau) = \sum_{m} e^{2u_j m_j + \tau_{jk} m_j m_k} ; \tag{14}$$

where the summation convention as to repeated indices is applied for $j, k = 1, 2, \cdots, p$; and where Σ_m denotes a p-fold sum as $m_1, m_2, \cdots, m_p$ vary independently over all integer values from $-\infty$ to $+\infty$.

$\rho = (\rho_1, \rho_2, \cdots, \rho_p)$ denotes any zero of the θ-function: $\theta(\rho; \tau) = 0$. Most conveniently, ρ may be taken equal to half an odd period of the θ-function:

$$\rho_k = \tfrac{1}{2}\,(\alpha_k + \beta_i \tau_{ik}) ,$$

where α_j, β_j are any $2p$ integers such that $\alpha_j \beta_j$ is odd.

Having thus completely described $A(g; \mu)$, we now define

$$m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = \textit{lower bound } A(g; \mu), \tag{15}$$

when the Riemann surface R varies over all of its type, while g varies over all sense-preserving topological correspondences between the given contours and the curves of transition of R.

This definition of m_r, while not as concrete as the one in terms of areas previously given (formula (1)), is more pertinent for the method of investigation employed, and the only one which we actually use. It is a fact that the two definitions of m_r agree,[9] but we shall not have need of this in the present work.

[9] Provable by the Lebesgue definition of the area of a surface: $S(\Sigma) = \lim \inf S(P)$, where P is any polyhedral surface approaching to Σ, and by the conformal mapping of P on an appropriate Riemann surface. See the paper cited in footnote 1, p. 318.

The inequalities (3), (4) obeyed by m_r can be deduced from the present definition (15) by considering the different modes of degeneration of the Riemann surface R.

Turning now to the case of a one-sided minimal surface $\bar{M}_r$, we employ a two-sided covering surface in the form of a semi Riemann surface R' with an inversely conformal transformation T into itself, where T associates the points of R' in pairs, and leaves no point of R' fixed. Each pair of associated points of R' will correspond to a single point of $\bar{M}_r$.[10] This one-two correspondence between $\bar{M}_r$ and R' implies that the Betti number of R' is $2r - 2$ and that R' has $2k$ boundaries formed of k pairs of T-equivalent curves $C_1, C_1'; C_2, C_2'; \cdots; C_k, C_k'$. The genus of the complete Riemann surface R, of which R' is one of the symmetric halves, and (C_a, C_a') the curves of transition, is

$$p = 2r + 2k - 3.$$

In any parametric representation of $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$, each point of Γ_a corresponds to a pair of T-equivalent points on C_a, C_a' respectively. The path of integration C in the definition (10) of $A(g; \mu)$ consists of one of each pair of T-equivalent curves C_a, C_a': $C = C_1 + C_2 + \cdots + C_k$.

This being understood, we define:

$$\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = \textit{lower bound } A(g; \mu), \tag{16}$$

when the Riemann surface R varies over all surfaces of its kind, as described above, and g varies over all parametric representations of the given contours.

The inequalities (8), (8'), (9), (9'), governing $\bar{m}_r$, can be deduced from the definition (16), as well as from the previous one (7), based on areas, by considering the different modes of degeneration of R.

5. Before we can state our results, we need the definition of a minimal surface M_r or $\bar{M}_r$ bounded by $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$.

First, a two-sided surface M_r. Let R' be any semi Riemann surface of the topological form prescribed for M_r: Betti number r and k boundaries $C_1, C_2, \cdots, C_k$. Let $F_i(w)$ denote n analytic functions,

[10] This mode of representation of a one-sided surface is given by F. Klein, loc. cit., §23, p. 73.

defined and regular at every point of R', and obeying identically the relation

$$\sum_{i=1}^{n} F_i'^2(w) = 0. \tag{17}$$

The functions $F_i(w)$ need not be uniform on R', but may have, at most, pure imaginary periods. Consequently, the real parts of these functions, $\Re F_i(w)$, are uniform and regular on R'. Then the formulas

$$x_i = \Re F_i(w) \qquad (i = 1, 2, \cdots, n) \tag{18}$$

define a minimal surface M_r; and M_r will be bounded by $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ if the harmonic functions (18) attach continuously to boundary values on $C_1, C_2, \cdots, C_k$:

$$x_i = g_{\alpha i}(z), \tag{19}$$

which, for $\alpha = 1, 2, \cdots, k$, are parametric representations of

$$\Gamma_1, \Gamma_2, \cdots, \Gamma_k,$$

respectively.

In case the required minimal surface is one-sided, $\bar{M}_r$, we employ a semi Riemann surface R' with an inverse conformal automorphism T, of period two, and without fixed points. The topological form of R' must be that of a covering surface for $\bar{M}_r$, where each point of $\bar{M}_r$ corresponds to a pair of T-equivalent points of R'.

The definition of $\bar{M}_r$ is by the formulas (17), (18), (19), with the additional stipulation that, if w_1, w_2 denote any two T-equivalent points of R', then

$$\Re F_i(w_1) = \Re F_i(w_2) \qquad (i = 1, 2, \cdots, n). \tag{20}$$

Therefore, according to (18), the points w_1, w_2 of R' give the same point of $\bar{M}_r$.

The formulas (17), (18) are the same as the classic ones given by Weierstrass for $n = 3$ and a simply-connected minimal surface. So defined, the minimal surface appears in conformal representation on R'.

6. Our first result refers to two-sided minimal surfaces.

THEOREM I. *Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be any given contours, and r any even integer $\geqq 0$, where*

$$m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < +\infty \text{ (i.e., finite).} \tag{a}$$

Suppose that

(b) $m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < m_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + m_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j'')$

for every partition of the given contours:

$$(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = (\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + (\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

and every partition of r into even components:

$$r = r' + r''.$$

Suppose also that (unless $r = 0$)

(c) $$m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < m_{r-2}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k).$$

Then there exists a two-sided minimal surface M_r with Betti number r, bounded by $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$.

The area of M_r is $m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k)$.

The effect of postulating the strong form of inequality, $<$, in conditions (b), (c)—instead of the weak form $\leqq$, which, according to formulas (3), (4), holds in any case—is to prevent the degeneration of M_r into a number of component minimal surfaces bounded respectively by partial groups of the given contours, or into a minimal surface of lower topological structure.

A necessary and sufficient condition for the requirement (a) is that each contour be capable of bounding a simple-connected surface of finite area: $m_0(\Gamma_1) < +\infty$, $m_0(\Gamma_2) < +\infty, \cdots, m_0(\Gamma_k) < +\infty$. For this in turn, a sufficient condition is that each contour be rectifiable.

Our second result refers to one-sided minimal surfaces.

THEOREM II. *Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be any given contours, and r any integer, odd or even, $\geqq 0$, where*

(a) $$\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < +\infty \qquad (i.e., \textit{finite}).$$

Suppose that

(b) $\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < \bar{m}_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + \bar{m}_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j'')$,

(b′) $\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < \bar{m}_{r'}(\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + m_{r''}(\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j'')$

for every partition of the given contours:

$$(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = (\Gamma_1', \Gamma_2', \cdots, \Gamma_i') + (\Gamma_1'', \Gamma_2'', \cdots, \Gamma_j''),$$

and every partition of r:

$$r = r' + r''$$

into components, odd or even, except that in (b′), r'' must be even.

Suppose also that

(c) $$\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < \bar{m}_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

and, in case r is odd,

(c′) $$\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) < m_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k).$$

Then there exists a one-sided minimal surface $\bar{M}_r$ with Betti number r, bounded by $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$.

The area of $\bar{M}_r$ is $\bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k)$.

7. It was an important feature of our previous work on the Plateau problem that in the special case $n = 2$ it gave us the classic theorems of Riemann and Osgood-Carathéodory concerning the conformal mapping of plane regions. The same continues to be the case in the present work, and we obtain for $n = 2$ the classic theorem of Schottky[11] concerning the conformal mapping of multiply-connected plane regions, as well as a supplementary result relating to the correspondence between boundaries.

As a form of the symmetric Riemann surface R, we here find it most convenient to use that of a real algebraic curve $G(x, y) = 0$ (real coefficients), and for the inverse conformal automorphism of R, the interchange of conjugate complex points of the curve, $w = (x, y)$ and $\bar{w} = (\bar{x}, \bar{y})$. The curves of transition, formed of the points such that $w = \bar{w}$, are evidently the real branches of the algebraic curve. These divide R into two semi Riemann surfaces R', R''.

Our third result is the theorem of Schottky, re-proved by means of the functional $A(g; \mu)$.

THEOREM III. *Let P denote a plane region bounded by any k Jordan curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ without common points.*

Then there exists a conformal map of P on the semi Riemann surface R′ of a certain real algebraic curve R, of genus $k - 1$ and with k real branches $C_1, C_2, \cdots, C_k$.

This conformal map attaches continuously to a topological correspondence between $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ and $C_1, C_2, \cdots, C_k$.

The necessary and sufficient condition that two given plane regions P, Q be conformally equivalent is that the corresponding algebraic curves R, S be birationally equivalent.

[11] F. Schottky, "Ueber die conforme Abbildung mehrfach zusammenhängender ebener Flächen," *Crelle*, v. 83 (1877), pp. 300-351.

8. We conclude with the following remarks.

(i) Along the lines laid down in the papers cited in footnotes 1, 2 and 3, §19, §11 and §7, respectively, it is possible to remove the restrictive condition (a) in theorems I and II, and to obtain results relative to the case where the contours are any Jordan curves not intersecting one another, with, in general, m_r and $\bar{m}_r = +\infty$.

(ii) Some of the contours may be allowed to have a single point in common, as in the author's paper "Crescent-shaped minimal surfaces,"[12] in which this is done for two contours and $r = 0$.

(iii) The theory of minimal surfaces here presented extends to calculus of variations problems of very general form, where we seek extremal surfaces of a prescribed topological structure with a number of given boundaries. The existence of such an extremal surface will be determined by criteria analogous to the conditions (b), (c) in theorem I and conditions (b), (b′), (c), (c′) in theorem II.

[12] *Proc. Nat. Acad. Sci.*, v. 19 (1933), pp. 192–199.

A NOTE ON ANALYTIC ROTATIONS IN EUCLIDEAN SPACE OF FOUR DIMENSIONS

By A. E. Staniland*

In the theory of functions of a complex variable, when a function of x and y of the form $w = u(x, y) + iv(x, y)$ is given such that u and v are real functions of x and y possessing continuous first partial derivatives with respect to x and y and, moreover, satisfying the two equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

w is an analytic function of $z = x + iy$. Let us take $z = x_1 + ix_2$ and $w = x_3 + ix_4$, and let $w = F(z)$ be analytic. Subject the variables x_i $(i = 1, 2, 3, 4)$ to the transformation

$$x_i = \sum a_{ij} x'_j$$

which is to be regarded as a rotation in S_4 of the four mutually perpendicular coordinate axes Ox_i, that is,

$$\sum a_{ij} a_{kj} = \epsilon_{ik} \tag{1}$$

$$\epsilon_{ik} = 1 \text{ for } i = k; \ \epsilon_{ik} = 0 \text{ for } i \neq k$$

$$|\, a_{ij} \,| = +1.$$

We wish to determine necessary and sufficient conditions upon the a_{ij} so that the transformation, considered as one in the complex plane of w and z will transform $w = F(z)$, which is analytic, into $w' = F'(z')$ which is also analytic.[1]

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[1] The real representation in S_4 of an analytic function $w = F(z)$ is called an analytic (or regular) surface; it is a minimal surface whose orthogonal projection on each of the coördinate z and w planes is a conformal representation of the surface itself. For a detailed discussion see Karl Kommerell, Riemannsche Flächen in ebenen Raum von vier Dimensionen. *Math. Annalen*, vol. 60, 1905.

Our problem, considered geometrically, is the determination of all rotations of S_4 that transform analytic surfaces into analytic surfaces. Such rotations are called analytic (or regular). J. S. Taylor used the term "regular," loc. cit.

Under (1) $w = F(z)$ becomes

$$\sum a_{2j}x'_j + i\sum a_{4j}x'_j = F\Big(\sum a_{1j}x'_j + i\sum a_{3j}x'_j\Big). \tag{2}$$

Insist, now, that

$$\frac{\partial x'_3}{\partial x'_1} = \frac{\partial x'_4}{\partial x'_2} \quad \text{and} \quad \frac{\partial x'_3}{\partial x'_2} = -\frac{\partial x'_4}{\partial x'_1} \tag{3}$$

From (2) we obtain

$$\begin{aligned}
\frac{\partial x'_3}{\partial x'_1} &= \frac{(a_{31} + ia_{41}) - \lambda(a_{11} + ia_{21})}{\lambda(a_{13} + ia_{23}) - (a_{33} + ia_{43})} \\
\frac{\partial x'_4}{\partial x'_2} &= \frac{(a_{32} + ia_{42}) - \lambda(a_{12} + ia_{22})}{\lambda(a_{14} + ia_{24}) - (a_{34} + ia_{44})} \\
\frac{\partial x'_3}{\partial x'_2} &= \frac{(a_{32} + ia_{42}) - \lambda(a_{12} + ia_{22})}{\lambda(a_{13} + ia_{23}) - (a_{33} + ia_{43})} \\
\frac{\partial x'_4}{\partial x'_1} &= \frac{(a_{31} + ia_{41}) - \lambda(a_{11} + ia_{21})}{\lambda(a_{14} + ia_{24}) - (a_{34} + ia_{44})}
\end{aligned}$$

where $\lambda = dF/dz \neq 0$.

Substituting these values into (3) and clearing of fractions, we obtain the two polynomials in λ:

$$\begin{aligned}
&(CF - BG)\lambda^2 + (AG + BH - CE - DF)\lambda + (DE - AH) = 0 \\
&(CB + FG)\lambda^2 - (AC + BD + GE + HF)\lambda + (DA + EH) = 0
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
A &= a_{31} + ia_{41} & E &= a_{32} + ia_{42} \\
B &= a_{11} + ia_{21} & F &= a_{12} + ia_{22} \\
C &= a_{13} + ia_{23} & G &= a_{14} + ia_{24} \\
D &= a_{33} + ia_{43} & H &= a_{34} + ia_{44}
\end{aligned}$$

Now λ is arbitrary since $F(z)$ was assumed to be any analytic function of z. Hence the coefficients in (4) vanish and, by separation of reals and imaginaries, we obtain:

$$a_{14}a_{31} - a_{24}a_{41} + a_{11}a_{34} - a_{21}a_{44} - a_{13}a_{32} + a_{23}a_{42} - a_{33}a_{13} + a_{43}a_{22} = 0$$

$$a_{24}a_{31} + a_{14}a_{41} + a_{11}a_{44} + a_{21}a_{34} - a_{13}a_{42} - a_{23}a_{32} - a_{33}a_{22} - a_{43}a_{12} = 0$$

$$a_{14}a_{32} - a_{24}a_{42} + a_{11}a_{33} - a_{22}a_{44} + a_{12}a_{31} - a_{23}a_{41} + a_{34}a_{12} - a_{43}a_{21} = 0$$

$$a_{14}a_{42} + a_{24}a_{32} + a_{12}a_{44} + a_{34}a_{22} + a_{23}a_{31} + a_{41}a_{13} + a_{11}a_{43} + a_{21}a_{33} = 0$$

$$
\begin{aligned}
&a_{23}a_{24} - a_{12}a_{14} - a_{11}a_{13} + a_{23}a_{21} = 0\\
&a_{12}a_{13} - a_{23}a_{23} - a_{11}a_{14} + a_{21}a_{24} = 0\\
&a_{23}a_{12} + a_{13}a_{22} - a_{14}a_{21} - a_{11}a_{24} = 0\\
&a_{14}a_{22} + a_{12}a_{24} + a_{11}a_{23} + a_{13}a_{21} = 0\\
&a_{42}a_{44} - a_{32}a_{34} + a_{41}a_{43} - a_{31}a_{33} = 0\\
&a_{33}a_{32} - a_{43}a_{42} - a_{31}a_{34} + a_{41}a_{44} = 0\\
&a_{33}a_{42} + a_{32}a_{43} - a_{41}a_{34} - a_{31}a_{44} = 0\\
&a_{34}a_{42} + a_{32}a_{44} + a_{33}a_{41} + a_{31}a_{43} = 0
\end{aligned}
\tag{5}
$$

Multiply the fifth, sixth, seventh and eighth equations of (5) by $-a_{14}$, a_{13}, a_{23}, a_{24} respectively. Then add the first pair and the second pair to obtain two new equations. Next, choosing the multipliers in the order a_{23}, a_{24}, $-a_{14}$, a_{13}, repeat the process. We thus obtain, after grouping the terms, the set of four linear equations:

$$
\begin{aligned}
&a_{13}P + a_{21}R - a_{23}S = 0\\
&a_{12}Q + a_{21}R + a_{22}S = 0\\
&a_{21}Q + a_{12}R - a_{11}S = 0\\
&a_{21}P + a_{12}R + a_{11}S = 0
\end{aligned}
\tag{6}
$$

where $P \equiv a_{13}^2 + a_{14}^2 \qquad Q \equiv a_{23}^2 + a_{24}^2$

$R \equiv a_{24}a_{13} - a_{14}a_{23} \qquad S \equiv a_{13}a_{23} + a_{14}a_{24}$

Adding the first and second of equations (6) and likewise the third and fourth, we obtain

$$a_{13}(P + Q) + 2a_{21}R = 0$$

$$a_{21}(P + Q) + 2a_{12}R = 0$$

Case 1. $P + Q \neq 0 \qquad a_{12}^2 - a_{21}^2 = 0 \qquad a_{21} = \pm a_{12}$

Case 2. $P + Q = 0$, that is, $a_{13} = a_{14} = a_{23} = a_{24} = 0$ and, according to (1),

$$a_{21} = \pm a_{12} \text{ and } a_{22} = \pm a_{11}.$$

Hence in all cases

$$a_{21} = \pm a_{12}.$$

Similarly, by using the two sequences of multipliers:

$$-a_{13},\ -a_{14},\ -a_{24},\ a_{23} \quad \text{and} \quad a_{24},\ -a_{23},\ a_{13},\ a_{14}$$

we obtain

$$\begin{aligned} a_{11}P - a_{22}R - a_{21}S &= 0, \\ a_{11}Q - a_{23}R + a_{21}S &= 0, \\ a_{23}Q - a_{11}R - a_{13}S &= 0, \\ a_{23}P - a_{11}R + a_{13}S &= 0, \end{aligned}$$

whence

$$\begin{aligned} a_{11}(P + Q) - 2a_{23}R &= 0, \\ a_{23}(P + Q) - 2a_{11}R &= 0, \end{aligned}$$

and, if $P + Q \neq 0$,

$$a_{23} = \pm a_{11};$$

but if $P + Q = 0$, then by (1),

$$a_{22} = \pm a_{11} \text{ and } a_{21} = \pm a_{12}.$$

Hence in all cases

$$a_{22} = \pm a_{11}.$$

Note that P, Q, S and R are respectively the sum of squares of the elements of the first row, the sum of squares of the elements of the second row, the inner product of the two rows, and the expansion of the minor determinant

$$\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}.$$

The method of procedure is now clear and from these same four equations we can show, mutatis mutandis, that $a_{23} = \pm a_{14}$ and $a_{24} = \pm a_{13}$. Likewise, by means of the last four equations of (5), we can prove that

$$a_{41} = \pm a_{32};\ a_{42} = \pm a_{31};\ a_{43} = \pm a_{34};\ a_{44} = \pm a_{33}.$$

Finally, equations (5), which must be satisfied identically, may be used to determine the correct association of signs. We thus obtain the two types of solutions:[2]

Type A

$$a_{22} = a_{11} \qquad a_{21} = -a_{12}$$
$$a_{24} = a_{13} \qquad a_{23} = -a_{14}$$
$$a_{42} = a_{31} \qquad a_{41} = -a_{32}$$
$$a_{44} = a_{33} \qquad a_{43} = -a_{34}$$

Type B

$$a_{22} = -a_{11} \qquad a_{21} = a_{12}$$
$$a_{24} = -a_{13} \qquad a_{23} = a_{14}$$
$$a_{42} = -a_{31} \qquad a_{41} = a_{32}$$
$$a_{44} = -a_{33} \qquad a_{43} = a_{34}$$

These types are sufficient as well as necessary solutions of equations (5), as may easily be verified. It should be noted that for both forms of solution, the value of the determinant $|a_{ij}|$ is necessarily $+1$, so that in both cases we are dealing with directly orthogonal transformations. In fact,

$$|a_{ij}| = L_1L_2 - 0_1^2 - 0_2^2 = 1,$$

where

$$L_1 \equiv a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 = 1$$
$$L_2 \equiv a_{31}^2 + a_{32}^2 + a_{33}^2 + a_{34}^2 = 1$$
$$0_1 \equiv a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} + a_{14}a_{34} = 0$$
$$0_2 \equiv a_{11}a_{32} - a_{12}a_{31} + a_{13}a_{34} - a_{14}a_{33} = 0.$$

In the case of type A, the rotations of S_4 may be expressed in the plane of z and w by the transformation:

$$z = A_{11}z' + A_{12}w'$$
(7)
$$w = A_{21}z' + A_{22}w'$$

[2] These two forms of solution were obtained by J. S. Taylor in A Four-space Representation of Complex Plane Analytics, M. I. T. *Journal of Mathematics and Physics*, vol. II, no. 1, Dec. 1922. His proof of their necessity was obtained by a geometric argument.

where

$$z' = x_1' + ix_2', \qquad w' = x_3' + ix_4',$$
$$A_{11} = a_{11} - ia_{12}, \qquad A_{12} = a_{13} - ia_{14},$$
$$A_{21} = a_{21} - ia_{32}, \qquad A_{22} = a_{33} - ia_{34},$$

and by (1)

$$A_{11}\bar{A}_{11} + A_{12}\bar{A}_{12} = A_{21}\bar{A}_{21} + A_{22}\bar{A}_{22} = 1,$$
$$A_{11}\bar{A}_{21} + A_{12}\bar{A}_{22} = 0, \mid A_{ij} \mid = e^{i\varphi}, \varphi \text{ real}.$$

For type B, the rotation is expressible in the z, w plane as the transformation:

$$z = \bar{A}_{11}\bar{z}' + \bar{A}_{12}\bar{w}'$$
$$w = \bar{A}_{21}\bar{z}' + \bar{A}_{22}\bar{w}' \tag{8}$$

or, alternatively, in the form

$$\bar{z} = A_{11}z' + A_{12}w'$$
$$\bar{w} = A_{21}z' + A_{22}w' \tag{8'}$$

the A_{ij} being the same as in (7).

It should be observed that the rotations of type A form a group; those of type B do not, since the product of any two such is of type A.

Rotations of Type A Carrying $z = 0$ into $w' = M_1 z'$. By means of the conditions on the coefficients A_{ij} of (7) we may express (7) in the form

$$z = A_{11}z' + A_{12}w'$$
$$w = -a\bar{A}_{12}z' + a\bar{A}_{11}w' \tag{7'}$$

where $a = e^{i\varphi}$ is the value of the determinant of (7). If (7') is to carry $z = 0$ into $w' = M_1 z'$, we must have

$$w' = -\frac{A_{11}}{A_{12}}z' = M_1 z',$$

so that $A_{11} = -A_{12}M_1$. The transformation then assumes the form

$$z = -A_{12}M_1 z' + A_{12}w'$$
$$T: \qquad w = -a\bar{A}_{12}z' - a\bar{M}_1\bar{A}_{12}w'$$
$$A_{12}\bar{A}_{12}(1 + M_1\bar{M}_1) = 1$$

It is easily seen that T carries $w = 0$ into $w' = -\frac{1}{\bar{M}_1} z'$, a result that was to be expected since the real representations in S_4 of $z = 0$ and $w = 0$ are two completely perpendicular coordinate planes, and the planes $w' = M_1 z'$ and $w' = -\frac{1}{\bar{M}_1} z'$ are also completely perpendicular.

Rotations of Type A Leaving a Given Plane Absolutely Invariant.

Consider next the transformation

$$R_1: \quad \begin{aligned} z &= e^{i\varphi} z' \\ w &= w', \end{aligned}$$

the effect of which, in S_4, is to leave the plane of x_3 and x_4 absolutely invariant while turning the plane of x_1 and x_2 upon itself through the angle φ_1. From T and R_1 we obtain

$$T^{-1}R_1T: \quad \begin{aligned} (1 + M_1\bar{M}_1)z &= (1 + M_1\bar{M}_1 e^{i\varphi_1})z' + \bar{M}_1(1 - e^{i\varphi_1})w' \\ (1 + M_1\bar{M}_1)w &= M_1(1 - e^{i\varphi_1})z' + (M_1\bar{M}_1 + e^{i\varphi_1})w', \end{aligned}$$

the effect of which is to leave the plane $w = -\frac{1}{\bar{M}_1} z$ absolutely fixed, rotating the plane $w = M_1 z$ upon itself through the angle φ_1.

The transformation

$$R_2: \quad \begin{aligned} z &= z' \\ w &= e^{i\varphi_2} w' \end{aligned}$$

leaves $z = 0$ absolutely fixed and turns $w = 0$ through the angle φ_2. From T and R_2 we obtain

$$T^{-1}R_2T: \quad \begin{aligned} (1 + M_1\bar{M}_1)z &= (M_1\bar{M}_1 + e^{i\varphi_2})z' + \bar{M}_1(e^{i\varphi_2} - 1)w' \\ (1 + M_1\bar{M}_1)w &= M_1(e^{i\varphi_2} - 1)z' + (1 + M_1\bar{M}_1 e^{i\varphi_2})w' \end{aligned}$$

under which the plane $w = M_1 z$ is absolutely invariant, and the plane $w = -\frac{1}{\bar{M}_1} z$ rotates upon itself through the angle φ_2. Each of the transformations $T^{-1}R_1T$ and $T^{-1}R_2T$ is a simple rotation in S_4.

The General Rotation of Type A. For $T^{-1}R_1T \cdot T^{-1}R_2T = T^{-1}R_2T \cdot T^{-1}R_1T$ we obtain

$$T^{-1}R_1R_2T: \quad \begin{aligned} (1 + M_1\bar{M}_1)z &= (e^{i\varphi_1}M_1\bar{M}_1 + e^{i\varphi_2})z' + \bar{M}_1(e^{i\varphi_2} - e^{i\varphi_1})w' \\ (1 + M_1\bar{M}_1)w &= M_1(e^{i\varphi_2} - e^{i\varphi_1})z' + (e^{i\varphi_1} + M_1\bar{M}_1 e^{i\varphi_2})w', \end{aligned}$$

which leaves $w = M_1 z$ and $w = -\frac{1}{\bar{M}_1} z$ invariant (not point for point) turning the former upon itself through the angle φ_1, the latter likewise through the angle φ_2. The general transformation in S_4 is a double rotation. The determinant of $T^{-1}R_1R_2T$ has the value $e^{i(\varphi_1+\varphi_2)}$. Since $T^{-1}R_1R_2T$ is the same transformation as (7), except in the outward form of its coefficients, we see that $\varphi = \varphi_1 + \varphi_2$. The form of $T^{-1}R_1R_2T$ shows that the group of ∞^4 rotations of type A consists of ∞^2 subgroups, of ∞^2 members each, corresponding to the ∞^2 possible choices of the complex parameter $M_1 (= m_1 + in_1)$. Each double rotation of a subgroup characterized by a fixed value of M_1 is seen to be the product in either order of two simple rotations $T^{-1}R_1T$ and $T^{-1}R_2T$, characterized by the angles φ_1 (with $\varphi_2 = 0$) and φ_2 (with $\varphi_1 = 0$), respectively, which give the extents of rotation in the invariant completely perpendicular planes determined by M_1.

We could also regard each of the ∞^2 subgroups as consisting of ∞^1 subgroups of ∞^1 members each. For we can fix φ_1 in ∞^1 ways, thus obtaining ∞^1 subgroups, the ∞^1 members of each corresponding to the ∞^1 values which φ_2 may assume with φ_1 fixed. We could, of course, fix φ_2 and allow φ_1 to vary, thus obtaining a similar classification.

A special case of the general transformation is of interest, namely, when $\varphi_2 = \varphi_1$. The coefficients are then independent of M_1 and $\bar{M}_1$ and all analytic planes through the origin are invariant.

Rotations of Type B. The transition to rotations of type B is immediate. We have, in fact, only to replace the unprimed z and w of $T^{-1}R_1R_2T$ by $\bar{z}$ and $\bar{w}$ respectively and the desired rotations are obtained. These rotations are distinguished from those of type A by the presence of two reflections, one accompanying each of the rotations in the invariant planes.

A Remark Concerning The Determinant, $|\, a_{ij} \,|$. The value of $|\, a_{ij} \,|$ may be shown explicitly in terms of φ, the amplitude of $|\, A_{ij} \,|$. For $|\, A_{ij} \,| = \cos\varphi + i \sin\varphi$, where

$$\cos\varphi = a_{11}a_{33} - a_{12}a_{34} - a_{13}a_{31} + a_{14}a_{32} \qquad \text{and}$$

$$\sin\varphi = a_{14}a_{31} + a_{13}a_{32} - a_{12}a_{33} - a_{11}a_{34}.$$

Expansion of $|\, a_{ij} \,|$ gives precisely

$$|\, a_{ij} \,| = \cos^2\varphi + \sin^2\varphi = 1.$$

ALMOST PERIODIC PROPERTIES OF BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS

By Robert H. Cameron

Introduction. In this paper we shall deal with the homogeneous linear system of differential equations

$$\frac{d\xi_\mu(t)}{dt} = \sum_{\nu=1}^{M} \alpha_{\mu,\nu}(t)\xi_\nu(t) \qquad (\nu = 1, 2, \cdots, n); \tag{1}$$

where the $\alpha_{\mu,\nu}(t)$ are real or complex a.p. (almost periodic) functions of the real variable t in the sense of Bohr (1). We shall let M be the least common module of the $\alpha_{\mu,\nu}(t)$, and shall assume that

$$\int_0^t \Re[\alpha_{1,1}(\tau) + \cdots + \alpha_{n,n}(\tau)]\, d\tau$$

is bounded for all t. (Here $\Re$ denotes *real part*). Under these conditions we shall show that if all the solutions of (1) are bounded, then there exists a linearly independent set of solutions of (1) such that for each μ, $(\mu = 1, 2, \cdots n)$, the sum of the squares of the absolute values of the i *th* components of the solutions is a.p. with a module contained in M. Moreover under a somewhat stronger hypothesis we shall characterize the individual components of the above mentioned solution in terms of a.p. functions.

Notation. We shall use a second subscript on the letter ξ to distinguish between various solutions of (1); and in particular we shall let

$$[\xi_{1,1}(t), \xi_{2,1}(t) \cdots, \xi_{n,1}(t)], [\xi_{1,2}(t), \xi_{2,2}(t), \cdots, \xi_{n,2}(t)], \cdots, [\xi_{1,n}(t), \xi_{2,n}(t), \cdots, \xi_{n,n}(t)]$$

denote a fundamental (linearly independent) set of n solutions. Thus if $A(t)$ and $X(t)$ stand for the matrices of the $\alpha_{\mu,\nu}(t)$ and $\xi_{\mu,\nu}(t)$ respectively and $D(X)$ denotes the matrix of the derivatives $\frac{d}{dt}\xi_{\mu,\nu}(t)$, we may replace (1) by

$$D(X) = A \cdot X. \tag{1a}$$

Semi-normal functions.

DEFINITION. A numerical or vector function $f(t)$ will be called *semi-normal* if to each sequence of real numbers $h_1, h_2, \cdots$ there corresponds a subsequence $h_{n_1}, h_{n_2}, \cdots$, such that $\lim_{i\to\infty} f(t + h_{n_i})$ exists uniformly in t in every finite interval.

This definition differs from Bochner's (2) definition of a normal function in that the uniformity is postulated merely on every finite interval instead of on the whole infinite interval $(-\infty < t < +\infty)$ at once. Since the term *normal function* is synonymous with *a.p. function* it follows that every a.p. function is semi-normal. On the other hand if $A(t)$ is a.p., every bounded solution of (1a) is semi-normal; for out of a sequence $h_1, h_2, \cdots$ we can choose a subsequence $h_{n_1}, h_{n_2}, \cdots$ such that $\lim_{i\to\infty} A(t + h_{n_i})$ converges uniformly for all t to a function $A^*(t)$ while $\lim_{i\to\infty} X(h_{n_i})$ converges, and hence such that $\lim_{i\to\infty} X(t + h_{n_i})$ converges to a solution $X^*(t)$ of $D(X^*) = A^* \cdot X^*$ uniformly in every finite interval. Thus semi-normal functions play an important rôle in connection with linear differential equations with a.p. coefficients.

LEMMA 1. *Every bounded solution of a linear differential equation with constant coefficients is semi-normal.*

LEMMA 2. *If $F(x, y, z, \cdots, w)$ is uniformly continuous in all its arguments over the range of the semi-normal functions $f(t)$, $g(t)$, $h(t)$, $\cdots$, $k(t)$, then $F[f(t), g(t), h(t), \cdots, k(t)]$ is semi-normal.*

The first of these lemmas is merely a formal statement of the fact noted above and the second is an obvious consequence of the definition.

Translation transforms.

DEFINITION. If $f(t)$ is a numerical or vector function of the real variable t, and $h_1, h_2, \cdots$ is a sequence of real numbers such that

$$g(t) \equiv \lim_{i\to\infty} f(t + h_i)$$

exists uniformly in every finite interval, then $g(t)$ will be called a *translation transform* of $f(t)$ by $h_1, h_2, \cdots$. If in addition the sequence $h_1, h_2, \cdots$ is such that $\lim_{i\to\infty} \lambda h_i$ exists (mod 2π) for every element λ of a certain module M, then $g(t)$ will be called an *M-translation transform* of $f(t)$ by $h_1, h_2, \cdots$.

It is obvious that if $f(t)$ is an a.p. function with the module M, then each M-translation transform of $f(t)$ is an a.p. function, and the set of all of them is the closure set of the function.

LEMMA 3. *If $g(t)$ is the translation transform of $f(t)$ by $h_1, h_2, \cdots$ and $p(t)$ is the translation transform of $g(t)$ by $k_1, k_2, \cdots$, then $p(t)$ is the*

translation transform of $f(t)$ by some sequence $l_1, l_2, \cdots$. Moreover if $\lim_{i\to\infty} \lambda h_i$ *exists* (mod 2π) *for every element* λ *of a certain module* M, *the sequence* $l_1, l_2, \cdots$ *can be so chosen that for each* λ *of* M,

$$\lim_{i\to\infty} \lambda(h_i + k_i - l_i) = 0 \qquad (\text{mod } 2\pi).$$

For we can choose corresponding to each index i a positive integer $n_i > i$ such that $|f(t + h_{n_i} + k_i) - g(t + k_i)| < 1/i$ when $|t| < i$. It follows that $p(t)$ is the translation transform of $f(t)$ by

$$h_{n_1} + k_1, h_{n_2} + k_2, \cdots.$$

LEMMA 4. *A translation transform of a semi-normal function is semi-normal.*

For let $n_1, n_2, \cdots$ be a sequence of indices chosen as in Lemma 3, and by virtue of the semi-normality of $f(t)$ let E be an infinite set of positive integers such that $\lim_{i\to\infty \text{ over } E} f(t + h_{n_i} + k_i)$ exists uniformly on every finite interval. Then $\lim_{i\to\infty \text{ over } E} g(t + k_i)$ exists uniformly on every bounded interval and $g(t)$ is semi-normal.

LEMMA 5. *If $X(t)$ is a bounded solution of the equation* (1a) *in which $A(t)$ is a.p. with the module M, then the M-translation transform of $X(t)$ by a sequence $h_1, h_2, \cdots$ is a solution of the equation obtained by replacing $A(t)$ in* (1a) *by the M-translation transform of $A(t)$ by $h_1, h_2, \cdots$.*

This lemma is a consequence of the convergence properties of solutions of a sequence of differential equations.

Stationary functions.

DEFINITION. A numerical or vector function $f(t)$ will be called *stationary* with respect to a module M if for all t

$$\lim_{i\to\infty} f(h_i + t) = f(t) \tag{2}$$

for every sequence $h_1, h_2, \cdots$ which satisfies for each element λ of M,

$$\lim_{i\to\infty} \lambda h_i = 0 \qquad (\text{mod } 2\pi). \tag{3}$$

Note. The definition of a stationary solution given in this paper is the same as the definition given by the author in a paper in the Acta Mathematica (3), but is not the same as and not equivalent to his definition given in a paper in the Annals of Mathematics (4). The author wishes this definition to supercede that of the Annals paper.

LEMMA 6. *A function $f(t)$ will be stationary with respect to a module M if and only if out of every sequence $h_1, h_2, \cdots$ which satisfies* (3) *for each*

element λ of M it is possible to select a subsequence $h_1', h_2', \cdots$ such that (2) holds for all t. For if $f(t)$ is not stationary there is a sequence $h_1, h_2, \cdots$ satisfying (3) for all λ in M and a real number x_0 such that $|f(x_0 + h_i) - f(x_0)| > \epsilon$ for all i and some positive ϵ independent of i; and hence the condition of the lemma cannot hold. On the other hand if $f(t)$ is stationary it is obvious that the condition does hold.

Conditions under which a function is almost periodic.

We shall now give a theorem in terms of the above definition which is based on and may be used in place of Favard's method (5) of dealing with differential equations with a.p. coefficients. A lemma which is similar to this theorem and serves a similar purpose has already been given by the author in his Annals paper (4).

THEOREM I. *A necessary and sufficient condition that a (numerical or vector) function $f(t)$ be a.p. with a module contained in the module M is that it be continuous and semi-normal and have all of its M-translation transforms stationary with respect to M.*

To show that this theorem holds, we need only verify the sufficiency of the condition, since its necessity is obvious. Let us therefore assume that the condition holds but that $f(t)$ is not normal (a.p.) with a module contained in M. Then it follows that there exists a sequence $h_1, h_2, \cdots$ satisfying (3) for all λ in M and an $\epsilon > 0$ such that

$$\sup_t |f(t + h_i) - f(t)| > \epsilon$$

for all integers i. Let $t_1, t_2, \cdots$ be values of t such that

$$|f(t_i + h_i) - f(t_i)| > \epsilon \tag{4}$$

for all i, and by virtue of the semi-normality of $f(t)$ let E_1 be an infinite set of positive integers such that $\lim_{i \to \infty \text{ over } E_1} f(t + t_i)$ and $\lim_{i \to \infty \text{ over } E_1} f(t + t_i + h_i)$ exist uniformly in every bounded interval and such that $\lim_{i \to \infty \text{ over } E_1} \lambda t_i$ exists (mod 2π) for every element λ in M. Let the limits over E_1 of $f(t + t_i)$ and $f(t + t_i + h_i)$ be denoted by $f^*(t)$ and $f^{**}(t)$ respectively, and by virtue of Lemma 4, let E_2 be an infinite subset of E_1 such that $f^{***}(t) \equiv \lim_{i \to \infty \text{ over } E_2} f^*(t - t_i)$ exists uniformly on every finite interval. Then by Lemma 3, $f^{***}(t)$ is a translation transform of $f(t)$, and since $f(t)$ is stationary, $f^{***}(t) \equiv f(t)$. Hence $f^{**}(t)$ is a transform of $f^*(t)$ through $f(t)$, and since $f^*(t)$ is stationary, $f^*(t) \equiv f^{**}(t)$. Since this contradicts (4), the theorem is proved.

The translation matrices of a solution.

DEFINITION. Let $X(t)$ be a non-singular solution of the equation

(1a) in which $A(t)$ is a.p. with the module M. Then if $h_1, h_2, \cdots$ is a sequence such that (3) holds for every element λ of M and such that $\lim_{i\to\infty} X(h_i)$ exists, the matrix

$$[X(0)]^{-1}\Big[\lim_{i\to\infty} X(h_i)\Big]$$

is called the *translation matrix* of X with respect to $h_1, h_2, \cdots$.

The significance of this definition will be shown in the following lemma.

LEMMA 7. *Let $X(t)$ be a non-singular solution of the equation* (1a) *in which $A(t)$ is a.p. with the module M. Then if C is a translation matrix of X corresponding to a sequence $h_1, h_2, \cdots$, it follows that*

$$[X(t)]\cdot C = \lim_{i\to\infty} X(t + h_i)$$

uniformly in every finite interval.

For it follows from the convergence properties of solutions of sequences of differential equations that the above limit exists uniformly for all t, and is a solution of (1a), since $\lim_{i\to\infty} A(t + h_i) = A(t)$ uniformly in t. It follows that the limit function is equal to $X(t)$ multiplied on the right by a constant matrix. Since C satisfies the equation when $t = 0$, it satisfies it for all t, and the lemma holds.

We shall now consider the whole set of translation matrices and shall show that they form a group if a simple condition on the Wronskian is fulfilled.

THEOREM II. *Let $X(t)$ be a non-singular solution of* (1a) *in which $A(t)$ is a.p. with the module M, and let $\mathfrak{G}$ be the set of translation matrices of X. Then if $X(t)$ and $\int_0^t \Re[\alpha_{1,1}(\tau) + \cdots + \alpha_{n,n}(\tau)]\,d\tau$ are bounded, it follows that $\mathfrak{G}$ is closed and forms a group under matrix multiplication.*

The fact that $\mathfrak{G}$ is closed does not depend on the hypothesis of boundedness on $X(t)$ and the integral. For if $C^{(1)}, C^{(2)}, \cdots$ are elements of $\mathfrak{G}$ which converge to a matrix C and $h_1^{(i)}, h_2^{(i)}, \cdots$ is a sequence with respect to which $C^{(i)}$ is a translation matrix ($i = 1, 2, \cdots$), it follows that we can choose a sequence of positive integers $n_1, n_2, \cdots$ such that $\| [X(0)]C^{(i)} - X(h_{n_i}^{(i)}) \| < 1/i$ and such that $\| A(t) - A(t + h_{n_i}^{(i)}) \| < 1/i$. Here $\| K \|$ denotes the greatest of $| k_{\rho,\sigma} |$. Thus C is a translation matrix of X corresponding to $h_{n_1}^{(1)}, h_{n_2}^{(2)}, \cdots$.

To verify the fact that $\mathfrak{G}$ is a group, we first note that Lemma 3 implies that if C_1 and C_2 are in $\mathfrak{G}$, $C_1\cdot C_2$ is also in $\mathfrak{G}$. Moreover $\mathfrak{G}$ contains the identity which corresponds to $0, 0, \cdots$. Finally, the fact

that $\mathfrak{G}$ contains inverses follows from a lemma proved by the author in a paper in the Annals (4). This lemma shows that under the hypothesis of the present theorem, every element of $\mathfrak{G}$ is equivalent to a matrix in the diagonal form with multipliers having the absolute value unity. But the inverse of such a matrix can be expressed as the limit of a sequence of its positive powers; and since $\mathfrak{G}$ is closed and contains all the positive powers of its elements, it must contain their inverses.

The translation matrix group. Theorem II now justifies us in calling the set of translation matrices of a solution X of (1a) *the translation matrix group of* X whenever the hypotheses of Theorem II are satisfied. Moreover we shall now apply to it a theorem of von Neumann (6) on matrix representations of groups.

THEOREM III. *Let $A(t)$ be an n-by-n a.p. matrix function such that $\int_0^t \Re[\alpha_{1,1}(\tau) + \cdots + \alpha_{n,n}(\tau)]\, d\tau$ is bounded, and let $X(t)$ be a bounded non-singular solution of* (1a) *and $\mathfrak{G}$ its translation matrix group. Then there exists a non-singular constant matrix K independent of C which takes each matrix C of $\mathfrak{G}$ into a unitary matrix $K^{-1}CK$. Thus the translation matrix group of $X(t)\cdot K$ consists entirely of unitary matrices.*

For the above mentioned theorem of von Neumann states that a bounded representation of a group is equivalent to a unitary representation; and $\mathfrak{G}$ is obviously a bounded representation of itself. Thus there exists a non-singular matrix K which takes every element C of $\mathfrak{G}$ into a unitary matrix $K^{-1}CK$.

We shall now prove the invariance of translation matrix groups under limiting translations.

LEMMA 8. *Let $A(t)$ be an n-by-n a.p. matrix function such that $\int_0^t \Re[\alpha_{1,1}(\tau) + \cdots + \alpha_{n,n}(\tau)]\, d\tau$ is bounded, and let M be the module of A, and $X(t)$ a bounded solution of* (1a) *having the translation matrix group $\mathfrak{G}$. Then $\mathfrak{G}$ is also the translation matrix group of every M-translation transform of X.*

For if X' is the M-translation transform of X by a sequence $h_1, h_2, \cdots$, then X' is transformed back into a solution of (1a) by some subsequence $-h_1', -h_2', \cdots$ of $-h_1, -h_2, \cdots$; say into XC. Then C is an element of $\mathfrak{G}$ and there is a sequence $k_1, k_2, \cdots$ which takes X into XC^{-1} and leaves A invariant. Moreover there is a sequence which effects the transformation which is the succession of the transformations effected by $-h_1', -h_2', \cdots$ and $k_1, k_2, \cdots$; and this sequence $l_1, l_2, \cdots$ transforms X' into X. Thus if $s_1, s_2, \cdots$ corresponds to an element C_1 of

$\mathfrak{G}$, it follows that X' is taken into $X'C_1$ by the succession of l_i, s_i, and h_i; and of course all this can be done by one sequence. Similarly if C_2 is a translation matrix of X' with respect to the sequence $s_1', s_2', \cdots$, the succession h_i, s_i' and l_i takes X into XC_2. It follows that the set of translation matrices of X' is $\mathfrak{G}$ and the lemma holds.

Functions of the components of a solution which are almost periodic in t. We have now developed sufficient machinery to prove the theorems mentioned in the introduction.

THEOREM IV. *Let $A(t)$ be an n-by-n a.p. matrix function which has the module M and for which $\int_0^t \Re[a_{1,1}(\tau) + \cdots + a_{n,n}(\tau)]\,d\tau$ is bounded. Then if (1a) has any bounded non-singular solutions, it has at least one non-singular solution $X(t) = \{\xi_{\mu,\nu}(t)\}$ such that*

$$g(t) = \sum_{\sigma=1}^{n} \xi_{\mu,\sigma}(t)\bar{\xi}_{\nu,\sigma}(t+\tau) \tag{5}$$

is a.p. in t for each real number τ and each pair of indices μ, ν. In particular

$$f(t) = \sum_{\sigma=1}^{n} |\xi_{\mu,\sigma}(t)|^2 \tag{6}$$

is a.p. for each μ. Moreover each of these a.p. functions has its module contained in M.

The solution X in question is the one whose translation matrix group $\mathfrak{G}$ consists entirely of unitary matrices (see Theorem III). The proof of the fact that $g(t)$ and hence $f(t)$ when calculated for this X are a.p. with a module contained in M is based on Theorem I; and we note at once that our functions are continuous and also, by virtue of Lemmas 1 and 2, semi-normal. It remains to show that every M-translation transform $g^*(t)$ of $g(t)$ by a sequence $h_1, h_2, \cdots$ is stationary with respect to M, and this will be done by showing that it satisfies the hypothesis of Lemma 6.

In order to show this let $h_1', h_2', \cdots$ be a subsequence of $h_1, h_2, \cdots$ such that $X^*(t) \equiv \lim_{i\to\infty} X(t + h_i)$ exists uniformly for every finite interval. Then $g^*(t) \equiv \sum_{\sigma=1}^{n} \xi^*_{\mu,\sigma}(t)\bar{\xi}^*_{\nu,\sigma}(t+\tau)$. Now let $k_1, k_2, \cdots$ be a sequence such that $\lim_{i\to\infty} \lambda k_i = 0 \pmod{2\pi}$ for every λ in M. Since by Lemma 8 $\mathfrak{G}$ is the translation matrix group of X^* and X^* is semi-normal, there exists an element C of $\mathfrak{G}$ and a subsequence $k_1', k_2', \cdots$ of $k_1, k_2, \cdots$ such that $\lim_{i\to\infty} X^*(t + k_i') = X^*C$ uniformly in every

finite interval. Hence if $C = \{c_{\rho,\sigma}\}$, we have $\sum_{\sigma=1}^{n} c_{\rho,\sigma}\, \bar{c}_{\rho',\sigma} = \delta_{\rho,\rho'}$, and

$$\lim_{i\to\infty} g^*(t + k_i') = \sum_{\sigma=1}^{n} \left[\sum_{\rho=1}^{n} \xi^*_{\mu,\rho}(t)\, c_{\rho,\sigma} \right] \left[\sum_{\rho'=1}^{n} \bar{\xi}^*_{\nu,\rho'}(t+\tau)\, \bar{c}_{\rho',\sigma} \right]$$

$$= \sum_{\rho=1}^{n} \sum_{\rho'=1}^{n} \xi^*_{\mu,\rho}(t)\, \bar{\xi}^*_{\nu,\rho'}(t+\tau) \sum_{\sigma=1}^{n} c_{\rho,\sigma}\, \bar{c}_{\rho',\sigma} = g^*(t)\,.$$

It follows that $g^*(t)$ is stationary with respect to M, and $g(t)$ is a.p. with a matrix contained in M.

COROLLARY. *Under the hypothesis of the theorem and for the same solution $X(t)$ the functions*

$$h(t) = \sum_{\sigma=1}^{n} \xi'_{\mu,\sigma}(t)\, \bar{\xi}_{\nu,\sigma}(t+\tau)$$

and

$$k(t) = \sum_{\sigma=1}^{n} \xi'_{\mu,\sigma}(t)\, \bar{\xi}'_{\nu,\sigma}(t+\tau)$$

are a.p. with modules contained in M.

For

$$h(t) = \sum_{\sigma=1}^{n} \left[\sum_{\rho=1}^{n} \alpha_{\mu,\rho}(t)\, \xi_{\rho,\sigma}(t) \right] \bar{\xi}_{\nu,\sigma}(t+\tau) = \sum_{\rho=1}^{n} \alpha_{\mu,\rho}(t)\, g_{\rho,\nu}(t)\,;$$

and a similar argument holds for $k(t)$.

The situation when $\mathfrak{G}$ is abelian.

Turning now to the case in which $\mathfrak{G}$ is abelian, we find that it is possible to characterize the $\xi_{\mu,\nu}(t)$ individually in terms of a.p. functions.

THEOREM V. *Let $A(t)$ be n-by-n a.p. matrix function which has the module M and for which $\int_0^t \Re\,[\alpha_{11}(\tau) + \cdots + \alpha_{n,n}(\tau)]\, d\tau$ is bounded; and let A be such that (1a) has a bounded non-singular solution whose translation matrix group is abelian. Then there exists a non-singular solution $X(t) = \{\xi_{\mu,\nu}(t)\}$ such that for each μ, ν,*

$$\xi_{\mu,\nu}(t) = f(t)\, e^{i \int \frac{g(t)}{[f(t)]^2}\, dt}$$

where $f(t) = |\, \xi_{\mu,\nu}(t)\,|$ and $g(t)$ are real a.p. functions having modules contained in M, and where the integral sign is used to denote a function whose derivative is equal to the integrand whenever the integrand exists.

The extra hypothesis concerning the abelian character of the transla-

tion matrix group of a solution implies that (1a) has a solution $X(t)$ whose translation matrix group $\mathfrak{G}$ consists entirely of matrices in the diagonal form; the diagonal elements having the absolute value unity. Thus the functions $|\xi_{\mu,\nu}(t)|$ and $\xi'_{\mu,\nu}(t)\,\bar{\xi}_{\mu,\nu}(t)$ are unchanged by limiting translation by a sequence $h_1, h_2, \cdots$ which takes $A(t)$ into itself; and the same is true of their translation transforms. It follows that these functions, which are obviously continuous and semi-normal, are a.p. with modules contained in M. This completes the proof of the theorem, since

$$\frac{d}{dt}\log\left[\frac{\xi_{\mu,\nu}(t)}{|\xi_{\mu,\nu}(t)|}\right] = \frac{ig(t)}{[f(t)]^2}$$

whenever $f(t) \neq 0$ if we let $f(t) = |\xi_{\mu,\nu}(t)|$ and $g(t) = (1/2i)$ $[\bar{\xi}_{\mu,\nu}(t)\,\xi'_{\mu,\nu}(t) - \xi_{\mu,\nu}(t)\,\bar{\xi}'_{\mu,\nu}(t)$.

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FRACTIONAL CALCULUS

By William Fabian

1. Introduction

Leibniz[1] was the first to give a meaning to a fractional differential coefficient of a function. Many writers[2] have investigated the subject since, and various definitions, mostly leading to the same form, have been given for the fractional integral and the fractional differential coefficient. The definition which we have adopted in a previous paper[3] is that of Riemann,[4] which is the one now usually taken. This definition of a λ^{th} integral, or a $(-\lambda)^{\text{th}}$ differential coefficient, of $f(z)$ along a simple curve l is

$$\left(\frac{d}{dz}\right)^{-\lambda} (l_a) f(z) = \frac{1}{\Gamma(\lambda + \gamma)} \left(\frac{d}{dz}\right)^{\gamma} \int_a^z (z - t)^{\lambda+\gamma-1} f(t)\, dt,$$

where λ may be real or complex, γ is the least integer greater than or equal to zero such that[5] $R(\lambda) + \gamma > 0$, and the integration and differentiation are along l. Riemann, Weyl[6] and others used this form for the real variable and for real values of λ. Hadamard[7] and Hardy and Littlewood[8] integrated fractionally along straight lines in the complex plane. Our extension of Riemann's definition, by which we are enabled to perform fractional integrations along any simple curve in the complex plane, has made it possible for us to study some properties of fractional contour integration.[9]

In the present paper we study some properties of the fractional integral at infinity, and deduce from our results a method of summability of series and integrals.

[1] Leibniz, *Opera ed. Dutens* 3, *Commercium philos. et Math.* (1695), 105.

[2] Post, Trans. Am. Math. Soc. (1930) vol. 32, p. 723–781, gives a comprehensive treatment with numerous references.

[3] Fabian, *Phil. Mag.* (to be published shortly).

[4] Riemann, *Gesammelte Werke* (1876), 331.

[5] $R(\lambda)$ stands for the real part of λ.

[6] Weyl, *Vierteljahrsschrift d. naturf. Ges. in Zurich*, 62 (1917), 296.

[7] Hadamard, *Journal de Math.* (4), 8 (1892), 101.

[8] Hardy and Littlewood, *Math. Zeitschrift* 34 (1931), 403.

[9] Fabian, *Phil. Mag.*, (to be published shortly).

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2. FRACTIONAL INTEGRATION AND DIFFERENTIATION

We define $D^{-\lambda}\,(l_a)\,f(z)$ by

$$D^{-\lambda}\,(l_a)\,f(z) = \frac{D^{\gamma}}{\Gamma(\lambda+\gamma)}\int_a^z (z-t)^{\lambda+\gamma-1} f(t)\,dt\,,$$

where D stands for $\frac{d}{dz}$, λ may be real or complex, γ is the least integer greater than or equal to zero such that $R(\lambda)+\gamma>0$, and the integration and differentiation are along the simple curve l.

The following lemma will be required:

LEMMA. *Let $\phi(z)$ be bounded on the finite part of l, where l starts from the origin and extends to infinity. Let $\phi(z)$ be continuous on l, except possibly at a finite number of points on any finite arc of l. If*

$$\frac{\phi(z)}{z^{\mu}} \to A$$

along l as $|z| \to \infty$, where A is a constant and $R(\mu) \geqq 0$, then, for $R(k) > -1$,

$$\frac{1}{z^{k+\mu+1}}\int_0^z (z-t)^k\,\phi(t)\,dt \to A\,\frac{\Gamma(k+1)\,\Gamma(\mu+1)}{\Gamma(k+\mu+2)}$$

along l as $|z| \to \infty$; the integration being along l.

PROOF. On l put $\phi(t) = At^{\mu} + \phi_1(t)$, where $\phi_1(t) = o(t^{\mu})$ at infinity. Then

$$\int_0^z (z-t)^k\,\phi(t)dt = A\int_0^z (z-t)^k\,t^{\mu}\,dt + \int_0^z (z-t)^k\,\phi_1(t)\,dt$$

$$= A\,\frac{\Gamma(k+1)\,\Gamma(\mu+1)}{\Gamma(k+\mu+2)}\,z^{k+\mu+1} + \int_0^z (z-t)^k\,\phi_1(t)\,dt\,, \qquad (1)$$

the integration being along l. Let τ and z be two points on l (in the t-plane) such that, at and between τ and z on l,

$$|\,\phi_1(t)\,| < \epsilon\,|\,t^{\mu}\,|$$

for a given positive number ϵ. Since $\phi(t)$ is bounded on the finite part of l, therefore at and between τ and the origin on l, $|\,\phi_1(t)\,|$ is less than a finite number $N(\tau)$ which depends only on τ. Hence

$$\left| \int_0^z (z-t)^\lambda \phi_1(t)dt \right| \leqq \left| \int_0^\tau (z-t)^\lambda \phi_1(t)dt \right| + \left| \int_\tau^z (z-t)^\lambda \phi_1(t)dt \right|$$

$$< N(\tau) \int_0^\tau |(z-t)^\lambda||\,dt| + \epsilon \int_\tau^z |(z-t)^\lambda||t^\rho||\,dt|$$

$$\leqq N(\tau) \int_0^\tau |(z-t)^\lambda||\,dt| + \epsilon \int_0^z |(z-t)^\lambda||t^\rho||\,dt|$$

$$= O(|z^\lambda|) + o(|z^{\lambda+\rho+1}|) \tag{2}$$

for $|z| > |z_0|$, a fixed number; the integration being, of course, along l. From (1) and (2) the lemma follows.

Let us call the following conditions the "conditions ψ_1 for $f(z)$ with respect to λ":

I. $f(z)$ is bounded on the finite part of l, where l starts from a fixed point τ, passes through the origin and the finite lower limit a of integration for $D^{-\lambda}(l_a)\, f(z)$, and extends to infinity.

II. If $R(\lambda) > 0$, $f(z)$ is continuous on l, except possibly at a finite number of points on any finite arc of l.

III. If $R(\lambda) \leqq 0$, $f(z)$ is analytic on l except possibly at infinity, and $z^\sigma f\left(\frac{1}{z}\right)$, where σ is a given quantity such that $R(\sigma) \geqq \gamma$, is analytic within a circle of centre the origin.

THEOREM 1. *Let conditions ψ_1 be satisfied for $f(z)$ with respect to λ. If*

$$\frac{f(z)}{z^\sigma} \to A$$

along l as $|z| \to \infty$, where A is a constant and $R(\sigma) \geqq \gamma$, then

$$\lim_{|z|\to\infty} \frac{D^{-\lambda}(l_a)\, f(z)}{z^{\lambda+\sigma}} = A\, \frac{\Gamma(\sigma+1)}{\Gamma(\lambda+\sigma+1)}.$$

PROOF. By condition III of ψ_1 and the hypothesis it follows that

$$\frac{f^{(\gamma)}(z)}{z^{\sigma-\gamma}} \to A\, \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\gamma+1)}$$

along l as $|z| \to \infty$. By the lemma,

$$\frac{1}{z^{\lambda+\sigma}} \int_0^z (z-t)^{\lambda+\gamma-1} f^{(\gamma)}(t)\, dt \to A\, \frac{\Gamma(\lambda+\gamma)\,\Gamma(\sigma+1)}{\Gamma(\lambda+\sigma+1)} \tag{1}$$

along l as $|z| \to \infty$; the integration being along l. We have, on integrating $D^{-\lambda}(l_a)f(z)$ by parts γ times,

$$D^{-\lambda}(l_a)f(z) = \sum_{n=0}^{\gamma-1} \frac{f^{(n)}(a)}{\Gamma(\lambda+n+1)}(z-a)^{\lambda+n} + D^{-\lambda-\gamma}(l_a)f^{(\gamma)}(z),$$

where

$$\sum_{n=0}^{\gamma-1} \frac{f^{(n)}(a)}{\Gamma(\lambda+n+1)}(z-a)^{\lambda+n}$$

is to be omitted if $\gamma = 0$. Consequently, for $|z| > |z_0|$, a fixed number,

$$D^{-\lambda}(l_a)f(z) = O(z^{\lambda+\gamma-1}) + D^{-\lambda-\gamma}(l_0)f^{(\gamma)}(z). \tag{2}$$

From (1) and (2) the conclusion follows.

The following properties will be called the "conditions ψ_2 for $f(z)$ with respect to λ and μ":

I. $f(z)$ is bounded on the finite part of l, where l starts from a fixed point τ, passes through the origin and extends to infinity.

II. λ and μ are two quantities, real or complex, and $R(\mu) > 0$.

If $R(\lambda) > 0$, $f(z)$ is continuous on l, except possibly at a finite number of points on any finite arc of l. For other values of λ, $f(z)$ is analytic on l except possibly at infinity.

III. a and b are two fixed points on l.

If $R(\lambda) \geqq 0$, or if λ is a negative integer, or if $R(\lambda) < 0$ and $f(a) = f'(a) = \cdots = f^{(\gamma-1)}(a) = 0$, γ being the least integer greater than or equal to zero such that $R(\lambda)+\gamma > 0$, then a and b may be anywhere on l. Otherwise, a is at τ or between the origin and τ on l, and b is neither at nor between a and τ on l.

THEOREM 2. *Let the conditions ψ_2 be satisfied for $f(z)$ with respect to λ and μ.*

If

$$\lim_{|z|\to\infty} \frac{D^{-\lambda}(l_a)f(z)}{z^{\rho}} = A, \text{ a constant},$$

where $R(\rho) \geqq 0$, then

$$\lim_{|z|\to\infty} \frac{D^{-\mu}(l_b)D^{-\lambda}(l_a)f(z)}{z^{\mu+\rho}} = A\,\frac{\Gamma(\rho+1)}{\Gamma(\mu+\rho+1)}.$$

For, with this hypothesis, conditions ψ_1 are satisfied for $D^{-\lambda}(l_a)f(z)$ with respect to μ for the μ-integration $D^{-\mu}(l_b)$. Applying therefore Theorem 1 to $D^{-\lambda}(l_a)f(z)$, we obtain

$$\lim_{|z|\to\infty} \frac{D^{-\mu}(l_b)\,D^{-\lambda}(l_a)f(z)}{z^{\mu+\rho}} = A\,\frac{\Gamma(\rho+1)}{\Gamma(\mu+\rho+1)}.$$

THEOREM 3. *Let conditions ψ_1 be satisfied for $f(z)$ with respect to λ. If*

$$\lim_{|z|\to\infty} \frac{D^{-\lambda}(l_a)f(z)}{z^{\rho}} = A, \text{ a constant},$$

where $R(\rho) > R(\lambda) + \gamma - 1$, then, for any fixed point b on l,

$$\lim_{|z|\to\infty} \frac{D^{-\lambda}(l_b)f(z)}{z^{\rho}} = A.$$

PROOF. For $|z| > |z_0|$,

$$\begin{aligned} D^{-\lambda}(l_a)f(z) &= O(z^{\lambda+\gamma-1}) + D^{-\lambda-\gamma}(l_a)f^{(\gamma)}(z) \\ &= O(z^{\lambda+\gamma-1}) + D^{-\lambda-\gamma}(l_b)f^{(\gamma)}(z) \\ &= O(z^{\lambda+\gamma-1}) + D^{-\lambda}(l_b)f(z). \end{aligned}$$

The conclusion follows immediately.

THEOREM 4. *Let conditions ψ_2 be satisfied for $f(z)$ with respect to λ and $-\lambda$.*

Let $f(a) = f'(a) = \cdots = f^{(\gamma-1)}(a) = 0$, except that $f^{(\gamma-1)}(a)$ need not be zero if λ is a negative integer.

If

$$\lim_{|z|\to\infty} \frac{D^{-\lambda}(l_a)f(z)}{z^{\rho}} = A, \text{ a constant},$$

where $R(\rho) \geqq 0$, then

$$\frac{f(z)}{z^{\rho-\lambda}} \to A\,\frac{\Gamma(\rho+1)}{\Gamma(\rho-\lambda+1)}$$

along l as $|z| \to \infty$.

For, putting $b = a$ and $\mu = -\lambda$ in Theorem 2, we get

$$\lim_{|z|\to\infty} \frac{D^{\lambda}(l_a)\,D^{-\lambda}(l_a)f(z)}{z^{\rho-\lambda}} = A\,\frac{\Gamma(\rho+1)}{\Gamma(\rho-\lambda+1)};$$

and under the conditions of the hypothesis,[3]

$$D^{\lambda}(l_a)D^{-\lambda}(l_a)f(z) = f(z).$$

3. SUMMABILITY OF SERIES

The results, which we have obtained here, lead to a method of summing series.

Let $w_1, w_2, \cdots w_n, \cdots$ be a set of points in the w-plane, taken in this order on a simple curve L, which passes through the origin. Let us call the direction of L, as defined by this order of points, the positive direction of L, and the opposite direction we shall call the negative direction of L. If the variable point w moves along L from a given point z in the positive direction, we shall call the arc traversed by it, excepting z, the positive side of z on L. The other arc of L, excepting z, which would be traversed by w when moving from z along L in the negative direction, will be called the negative side of z on L.

Consider the series

$$u_1 + u_2 + \cdots + u_n + \cdots \infty,$$

where

$$u_1, u_2, \cdots u_n, \cdots$$

are finite and independent of w and z. Let n_z be that value of n for which w_{n+1} lies on the positive side of the point z on L, and w_n lies on the negative side of z on L or at z. Suppose that the sum of the first n_z terms of the series is $F(z)$. If the series converges to a sum S, we have by Theorem 1, for all values of λ such that $R(\lambda) > 0$, and for any fixed point a on L,

$$\Gamma(\lambda + 1). \lim_{|z| \to \infty} \frac{D^{-\lambda}(L_a) F(z)}{z^\lambda} = S,$$

where z moves along L in the positive direction. Hence we can give the following definition:[19]

If

$$\Gamma(\lambda + 1). \lim_{|z| \to \infty} \frac{D^{-\lambda}(L_a) F(z)}{z^\lambda} = T,$$

a finite number, where z moves along L in the positive direction, and $R(\lambda) > 0$, then we shall say that the series

$$u_1 + u_2 + \cdots + u_n + \cdots \infty$$

is summable (λ, L_a) to sum T.

[19] The method of summation of Riesz is included in the method given here.

From Theorem 2 we obtain,

THEOREM 5. *If a series is summable* (λ, L_a), *then it is summable* $(\lambda + \mu, L_a)$ *to the same sum for all values of* μ *such that* $R(\mu) > 0$.

From Theorem 3 we have,

THEOREM 6. *If a series is summable* (λ, L_a), *then it is summable* (λ, L_b) *to the same sum.*

4. SUMMABILITY OF INTEGRALS

We can also deduce from our results a method of summing integrals. The integrals which can be summed by this method may be fractional.

Let L be a simple curve starting from a fixed point in the z-plane, passing through the origin and extending to infinity.

Consider the integral

$$F(z) = D^{-a}(L_c)f(z),$$

where c is a fixed point on L, and $F(z)$ is continuous on any arc of L extending to infinity, which contains the origin and an arbitrary fixed point a.

If $\lim_{|z|\to\infty} F(z)$ exists as a finite number S, then we have by Theorem 1, for all values of λ such that $R(\lambda) > 0$,

$$\Gamma(\lambda + 1). \lim_{|z|\to\infty} \frac{D^{-\lambda}(L_a)F(z)}{z^\lambda} = S.$$

We can therefore give the following definition:

If

$$\Gamma(\lambda + 1). \lim_{|z|\to\infty} \frac{D^{-\lambda}(L_a)F(z)}{z^\lambda} = T,$$

a finite number, where $R(\lambda) > 0$, *then we shall say that the infinite integral* $F(\infty)$ *is summable* (λ, L_a) *to sum* T.

Theorem 2 gives us:

THEOREM 7. *If an integral is summable* (λ, L_a), *then it is summable* $(\lambda + \mu, L_a)$ *to the same sum for all values of* μ *such that* $R(\mu) > 0$.

Theorem 3 gives us:

THEOREM 8. *If an integral is summable* (λ, L_a), *then it is summable* (λ, L_b) *to the same sum.*

ON ENTIRE FUNCTIONS WITH GAPS

By Henry Malin

1. This paper will be devoted to the proof of the following three theorems:

Theorem 1. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be an entire function, and let the λ_n*'s be a set of integers arranged in increasing order. Let*

$$\lambda_{n+1} - \lambda_n \to \infty .$$

Then $f(z)$ *has in every angle, the same order and the same type as in the entire plane.*

Theorem 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be an entire function of infinite order. Let the set $\{\lambda_n\}$ *be defined as in Theorem* 1. *Then every line from the origin is a line of Julia.*

Ritt[1] has defined the notion of order as applied to an entire function $f(s)$ which is represented by an everywhere absolutely convergent Dirichlet series. Order is by definition the

$$\overline{\lim_{\sigma \to -\infty}} \{(-\sigma)^{-1} \log \log M_f(\sigma)\},$$

where $M_f(\sigma)$ is the least upper bound of $|f(s)|$ on the vertical line with abscissae σ. Ritt has shown that if $f(s) = \sum_0^{\infty} a_n e^{-\lambda_n s}$ is an everywhere absolutely convergent series, the order of $f(s)$ is ρ when and only when

$$\overline{\lim_{n \to \infty}} \{(\lambda_n \log \lambda_n)^{-1} \log |a_n|\} = -\frac{1}{\rho}.$$

Ritt's result has as an immediate consequence, that the order of the integral of $f(s)$ is the same as the order of $f(s)$.

[1] Ritt, On Certain Points in the Theory of Dirichlet Series, American Journal of Mathematics, Vol. 50, p. 77, 1928.

The concluding theorem can now be stated.

Theorem 3. *Let*

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$$

be an entire everywhere (absolutely) covergent Dirichlet series. Let the λ_n*'s be a sequence of increasing numbers. Let*

$$\lambda_{n+1} - \lambda_n \to \infty .$$

Then, the order of $f(s)$ *in any strip, no matter how small, is the same as in the entire plane.*

2. In order to prove Theorems 1 and 3, the following lemma, which is a slight modification of a theorem due to Wiener,[2] will be found useful.

Lemma 1. *Let* $P(\theta) = \sum_0^N a_n e^{i\lambda_n \theta}$ *be a trigonometrical polynomial. Let the* λ_n*'s be arranged in increasing order. Let*

$$\lambda_{n+1} - \lambda_n \geqq L .$$

Then

$$\int_0^{2\pi} |P(\theta)|^2 \, d\theta \leqq \frac{16\pi^2}{\beta - \alpha} \int_\alpha^\beta |P(\theta)|^2 \, d\theta ,$$

where $\beta - \alpha = \dfrac{16\pi}{L}$.

From

$$\int_{-\infty}^{\infty} e^{iu\theta} \frac{\sin^2 \dfrac{L\theta}{2}}{\theta^2} \, d\theta = \begin{cases} \dfrac{\pi L}{2}\left(1 - \dfrac{|u|}{L}\right), & |u| < L \\ 0, & |u| > L , \end{cases}$$

it follows immediately that

$$\int_{-\infty}^{\infty} |P(\theta)|^2 \frac{\sin^2 \dfrac{L\theta}{2}}{\theta^2} \, d\theta = \frac{\pi L}{2} \sum_0^N |a_n|^2 . \tag{2.01}$$

Hence

$$\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} |P(\theta)|^2 \, d\theta \leqq \frac{\pi^2}{2L} \sum_0^N |a_n|^2 ,$$

[2] N. Wiener, A Class of Gap Theorems, Pisa Annali, Ser. 2, vol. 2, 1934, p. 367.

and in general

$$\int_{-\frac{\pi}{L}+\nu}^{\frac{\pi}{L}+\nu} |P(\theta)|^2\,d\theta \leqq \frac{\pi^3}{2L}\sum_0^N |a_n|^2. \tag{2.02}$$

Again, by (2.02), if $A > \frac{\pi}{L}$,

$$\int_{-A}^{A} |P(\theta)|^2\,d\theta \leqq \pi^2 A \sum_0^N |a_n|^2.$$

Thus

$$\left[\int_A^\infty + \int_{-\infty}^{-A}\right]\frac{|P(\theta)|^2}{\theta^2}\,d\theta = \int_A^\infty \frac{1}{\theta^2}\,d\int_{-\theta}^{\theta} |P(\theta)|^2\,d\theta$$

$$= -\frac{1}{A^2}\int_{-A}^{A} |P(\theta)|^2\,d\theta + 2\int_A^\infty \frac{d\theta}{\theta^2}\frac{1}{\theta}\int_{-\theta}^{\theta} |P(x)|^2\,dx$$

$$\leqq 2\pi^2 \sum_0^N |a_n|^2 \int_A^\infty \frac{dx}{x^2}$$

$$= \frac{2\pi^2}{A}\sum_0^N |a_n|^2.$$

Thus by (2.01)

$$\int_{-A}^{A} |P(\theta)|^2 \frac{\sin^2\frac{L\theta}{2}}{\theta^2}\,d\theta \geqq \left[\frac{\pi L}{2} - \frac{2\pi^2}{A}\right]\sum_0^N |a_n|^2.$$

Now let $A = \frac{8\pi}{L}$. Then

$$\int_{-\frac{8\pi}{L}}^{\frac{8\pi}{L}} |P(\theta)|^2\,d\theta \geqq \frac{4}{L^2}\int_{-\frac{8\pi}{L}}^{\frac{8\pi}{L}} |P(\theta)|^2 \frac{\sin^2\frac{L\theta}{2}}{\theta^2}\,d\theta$$

$$\geqq \frac{\pi}{L}\sum_0^N |a_n|^2 \tag{2.03}$$

If $\beta - \alpha = \frac{16\pi}{L}$, it follows that

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta |P(\theta)|^2\,d\theta \geqq \frac{1}{16}\sum_0^N |a_n|^2$$

$$\geqq \frac{1}{16\pi^3}\int_0^{2\pi} |P(\theta)|^2\,d\theta. \tag{2.04}$$

This proves the lemma. Also,

$$\int_{\alpha}^{\beta} |P(\theta)|^2 \, d\theta \geqq A \int_{\alpha+\gamma}^{\beta+\gamma} |P(\theta)|^2 \, d\theta . \tag{2.05}$$

3. Proof of Theorem 1.

Let the order of $f(z)$ be ρ. Then

$$|f(z)| \leqq e^{|z|^{\rho+\epsilon}}$$

for all ϵ.

Suppose that the order of $f(z)$ in the angle $\alpha \leqq \theta \leqq \beta$ is less than ρ; that is if $\alpha \leqq \theta \leqq \beta$

$$|f(z)| \leqq e^{|z|^{\rho-\delta}} \qquad \text{for } |z| > R, \tag{3.00}$$

where R is a sufficiently large number.

By (2.04)

$$\frac{16\pi^3}{\beta - \alpha} \int_{\alpha}^{\beta} |f(re^{i\theta})|^2 \, d\theta \geqq \int_{0}^{2\pi} |f(re^{i\theta})|^2 \, d\theta . \tag{3.01}$$

This result holds only for angles where

$$\beta - \alpha = \frac{16\pi}{L} .$$

Let it be noticed, however, that if we subtract an appropriate polynomial from $f(z)$, we can make the gaps between the remaining successive λ_n's large as we please. And thus $\beta - \alpha$ may be taken as small as we please.

By (3.01)

$$16\pi^3 e^{2|z|^{\rho-\delta}} \geqq \int_{0}^{2\pi} |f(re^{i\theta})|^2 \, d\theta . \tag{3.02}$$

Consider now

$$F(Z) = \int_{0}^{Z} f(\xi) \, d\xi$$

$$= \int_{0}^{re^{i\frac{\alpha+\beta}{2}}} f(\xi) \, d\xi + i \int_{\frac{\alpha+\beta}{2}}^{\theta} re^{i\theta} f(re^{i\theta}) \, d\theta ,$$

where Z is an arbitrary point in the plane.

Using the Schwarz inequality and (3.02)

$$|F(Z)| \leqq r^{1/2}\left\{\int_0^R \left|f\left(re^{i\frac{\alpha+\beta}{2}}\right)\right|^2 dr + \int_R^r \left|f\left(re^{i\frac{\alpha+\beta}{2}}\right)\right|^2 dr\right\}^{1/2}$$

$$+ r\sqrt{2\pi}\left(\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right)^{1/2}$$

$$\leqq r^{1/2}\left(K + \int_R^r e^{2\rho r^\rho - \delta} dr\right)^{1/2} + r\sqrt{2\pi}\sqrt{16\pi^2}\, e^{|a|r^\rho - \delta} \tag{3.03}$$

$$\leqq r^{1/2}\left(Ke^{2\rho r^\rho - \delta} + re^{2\rho r^\rho - \delta}\right)^{1/2} + 8\pi^2 r\, e^{r^\rho - \delta}$$

$$\leqq r^2 e^{r^\rho - \delta}$$

$$\leqq e^{r^\rho - \frac{\delta}{2}}.$$

It is well known that the integral of an entire function is itself an entire function and has the same order as the original function. Hence (3.03) contradicts the fact that $f(z)$ is of order ρ.

The same argument will carry the proof through for type and the theorem is proved.

Proof of Theorem 3.

Let the order of $f(\sigma + it)$ in the strip $b < t < a$ be less than ρ. Then

$$|f(\sigma + it)| < e^{\sigma^{-\sigma(\rho-\epsilon)}}, \qquad b < t < a, \qquad \sigma < -\sigma_0.$$

Consider now

$$F(\sigma + it) = \int_{s_1}^{s} f(\sigma + it)\, ds; \qquad s = \sigma + it, \qquad s_1 = \sigma_1 + it_1,$$

$$= \int_{\sigma_1}^{\sigma} f(\sigma + it_1)\, d\sigma + \int_{t_1}^{t} f(\sigma + it)\, dt,$$

where s_1, s are two arbitrary points in the strip

By the Schwarz inequality

$$|F(\sigma + it)| \leqq |\sigma_1 - \sigma|^{1/2}\left(\int_{\sigma_1}^{\sigma} |f(\sigma + it_1)|^2 d\sigma\right)^{1/2}$$

$$+ \text{const.}\left(\int_{t_1}^{t} |f(\sigma + it)|^2 dt\right)^{1/2} \tag{3.04}$$

$$\leqq |\sigma_1 - \sigma|\, e^{\sigma^{-\sigma(\rho-\epsilon)}} + \text{const.}\, e^{\sigma^{-\sigma(\rho-\epsilon)}}$$

$$\leqq |\sigma|\, e^{\sigma^{-\sigma(\rho-\epsilon)}} \qquad \text{for } \sigma < -\sigma_0.$$

Now let σ be an arbitrary vertical line. Let $M(\sigma)$ be the maximum modulus on the σ-line. Let t_σ be an ordinate for which $|f(\sigma+it_\sigma)| \geqq \frac{M(\sigma)}{2}$. Consider the strip S: $t_\sigma - \frac{a-b}{2} < t < t_\sigma + \frac{a+b}{2}$.

In S we have

$$|F(\sigma+it)| \geqq |F(\sigma+it_\sigma)| - \int_{t_\sigma}^{t} |f(\sigma+it)|\,dt$$

$$\geqq \frac{M(\sigma)}{2} - |b-a|^{1/2}\left(\int_{t_\sigma}^{t} |f(\sigma+it)|^2\,dt\right)^{1/2}$$

$$\geqq \frac{M(\sigma)}{2} - |b-a|^{1/2}\left(\int_{t_\sigma-\frac{a-b}{2}}^{t_\sigma+\frac{a-b}{2}} |f(\sigma+it)|^2\,dt\right)^{1/2}.$$

By (2.05), we have

$$\int_a^b |f(\sigma+it)|^2\,dt \geqq A\int_{a+l}^{b+l} |f(\sigma+it)|^2\,dt$$

Also

$$M(\sigma) \geqq Ce^{-\sigma\left(\rho-\frac{\epsilon}{2}\right)}.$$

Hence

$$|F(\sigma+it)| \geqq C_1 e^{-\sigma\left(\rho-\frac{\epsilon}{2}\right)} \qquad \text{for} \qquad t_\sigma - \frac{a-b}{2} < t < t_\sigma + \frac{a-b}{2}.$$

Or

$$\int_{t_\sigma-\frac{a-b}{2}}^{t_\sigma+\frac{a-b}{2}} |F(\sigma+it)|^2\,dt \geqq C_2 e^{-2\sigma\left(\rho-\frac{\epsilon}{2}\right)}.$$

And another application of (2.05) yields

$$\int_a^b |F(\sigma+it)|^2\,dt \geqq C_3 e^{-2\sigma\left(\rho-\frac{\epsilon}{2}\right)},$$

which contradicts (3.04) and thus the theorem is proved. We have used here again the remark made in Theorem 1, viz., an appropriate polynomial may be neglected in lemma 1 in order to make the strip as small as necessary.

Bieberbach has shown that if an entire function has infinite order along any line L, then L is a line of Julia. Theorem 1 holds for entire functions of infinite order. Combining Theorem 1 with Bieberbach's theorem, there results Theorem 2.

4. Polya[3] has shown that an entire function of coefficient density zero has in all angles, the same order and the same type as in the entire plane. Thus it is seen that Theorem 1 is not the most general of its kind.

Mandelbrojt and Gergen,[4] in a generalization to Dirichlet series of a theorem of Polya's[5] have proved Theorem 3. Their methods are more difficult than ours: it is necessary for them to use the concept of normal families and to introduce entire functions to prove the theorem.

[3] G. Polya, Lucken und Singularitaten von Potenzreihen, Math. Zeitsch., 1928–29, vol. 29, pp. 549–640.

[4] S. Mandelbrojt and J. J. Gergen, Functions defined by a Dirichlet Series, American Journal of Mathematics, vol. 53, 1931, pp. 1–14.

[5] This theorem states that an entire function of infinite order, with coefficient density zero, has all lines as lines of Julia. It is thus a more general theorem than Theorem 2.

EXISTENCE THEOREM ON IMPLICIT VECTOR FUNCTIONS

By I. M. Hostetter[1]

For those who consider a vector primarily as a function of two or more scalar variables varying independently, vector analysis is but a shorthand and there is no need for establishing existence theorems in the foundations of the subject, the existence theorems of scalar analysis sufficing. Recognition of a vector as an entity possessing properties other than magnitude immediately necessitates such proofs. One of the first to recognize this was E. Cunningham[2] who laid the foundations for such an attempt in 1912. Although this point of view has been adopted by many writers[3] on vector analysis little has since been done in the way of building up from the beginning a calculus of vectors as functions of one or more vector variables. We will here consider the fundamental theorem on the existence of a solution for any one of the vector variables $\mathbf{r}_i$ of the vector equation

$$\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m, \tau_1, \tau_2, \cdots, \tau_p) = 0,$$

where $\mathbf{F}$ is a vector function of the vector variables $\mathbf{r}_i$ and the scalar variables τ_j. The theorems on the existence of a solution for a system of scalar equations of vector and scalar variables and of a set of scalar equations of scalar variables follows immediately as a consequence of this theorem. For a history of the development of the proof of the implicit function theorem for scalar variables the reader is referred to the *Encyclopadie der Mathematischen Wissenschaften*[4] and for the various methods of proof to the very excellent treatise of Prof. Bliss.[5] Before stating the theorem a brief description of the notation to be used is necessary.

[1] Department of Mathematics, University of Washington.

[2] Theory of functions of a real variable, London M. S. Proc. 12, pp. 133–155.

[3] The following papers will be found interesting from this viewpoint: G. Y. Rainich, Tensor analysis without coordinates, Nat. Acad. Proc. 9, (1923), pp. 179–183, also American Journal 46, (1924), pp. 71–94; H. T. Flint, A generalized vector analysis of four dimensions, London R. S. Proc. (A) 103, (1923), pp. 644–663.

[4] II B I, §44 and footnote 30.

[5] The Princeton Colloquium, 1909, Fundamental existence theorems, by Prof. Gilbert Ames Bliss.

To indicate vectors bold-faced type will be used, and for scalars ordinary type. The scalar product

$$\overline{\mathbf{A}_1 \times \mathbf{A}_2} \,\vdots\, \mathbf{A}_3\mathbf{A}_4 \cdots \mathbf{A}_n = \{\cdots [(\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{A}_n) \cdot \mathbf{A}_{n-1}] \cdot \mathbf{A}_{n-2} \cdots \cdot \mathbf{A}_3\} \cdot \mathbf{A}_1$$

will be indicated by $[\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n]$, the cross product $\mathbf{A}_1 \times \mathbf{A}_2$ being interpreted as a scalar in 2-space, a vector in 3-space, and a skew symmetric polyadic of order $n-2$ in n-space, defined by

$$\mathbf{A}_1 \times \mathbf{A}_2 = e\,\epsilon_{i_1 i_2 \cdots i_n}\,\mathbf{a}_{i_1}\mathbf{a}_{i_2}\cdots\mathbf{a}_{i_n}\,|\,\mathbf{A}_1\,|\,|\,\mathbf{A}_2\,|\sin(\mathbf{A}_1, \mathbf{A}_2)\,,^6 \qquad (i_s = 1, \cdots, n-2)$$

where $(\mathbf{A}_1, \mathbf{A}_2)$ is the euclidean angle between $\mathbf{A}_1$ and $\mathbf{A}_2$ and $\mathbf{a}_{i_s}$ are unit vectors such that

$$\mathbf{a}_{i_s} \cdot \mathbf{a}_{i_m} = \delta^s_m\,, \qquad \mathbf{a}_{i_s} \cdot \mathbf{A}_1 = 0\,, \qquad \mathbf{a}_{i_s} \cdot \mathbf{A}_2 = 0\,,$$

and e has whichever of the two values 1 or -1 is necessary to make

$$[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_n \cdots \mathbf{A}_4\mathbf{A}_3] = e[\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n]\,.$$

Extending Gibbs' definition of the *third*[7] of a dyadic to n-space, we define the nth of a dyadic Φ by

$$\Phi_n = \{\cdots [(\Phi^{\times}_{\times}\Phi \,\vdots\, \Phi) \,\vdots\, \Phi] \,\vdots\, \Phi \cdots\} \,\vdots\, \Phi/n\,! \qquad \text{(a total of } n\ \Phi\text{'s)}$$

where the double dot and the double cross indicate that, after expansion of the product of the dyadic factors according to the distributive law, the resulting terms containing double dot and double cross products are to be expanded and interpreted in the same manner as the double dot product in the scalar product above. If $\Phi = \sum_i^n \mathbf{a}_i\mathbf{b}_i$, on expanding it follows at once that

$$\Phi'_n = [\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_n]\,[\mathbf{b}_1\mathbf{b}_2 \cdots \mathbf{b}_n]\,.$$

Let $\mathbf{r}_n$ be any real vector in n-space and represent by $(\nabla_{\mathbf{r}_n}\mathbf{F})_n$ the nth of the partial dyadic derivative $\nabla_{\mathbf{r}_n}\mathbf{F}$ of the vector function $\mathbf{F}$.

Then, *any vector equation* $\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m, \tau_1, \tau_2, \cdots, \tau_p) = 0$ *in the* m *vector variables* $\mathbf{r}_1, \cdots, \mathbf{r}_m$, *and the* p *scalar variables* $\tau_1, \cdots, \tau_p$, *may be solved for any one of the vector variables* $\mathbf{r}_i$ *in terms of the remaining variables,*

[6] The symbols $\epsilon_{i_1 i_2 \cdots i_n}$ and δ^s_m here employed have the usual significance of the tensor analysis, a repeated index indicating summation on that index.

[7] Gibbs-Wilson, Vector Analysis, p. 311.

$$\mathbf{r}_i = \mathbf{G}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_{i-1}, \mathbf{r}_{i+1}, \cdots, \mathbf{r}_m, \tau_1, \tau_2, \cdots, \tau_p),$$

provided that there exists a set of values

$$(\mathbf{r}_1)_0, (\mathbf{r}_2)_0, \cdots, (\mathbf{r}_m)_0, (\tau_1)_0, \cdots, (\tau_p)_0,$$

which satisfy the equation; that the function $\mathbf{F}$ *is continuous and possesses continuous first derivatives in the neighborhood of the values*

$$(\mathbf{r}_1)_0, \cdots, (\mathbf{r}_m)_0, (\tau_1)_0, \cdots, (\tau_p)_0;$$

and, finally, that $(\nabla_{\mathbf{r}_i}\mathbf{F})_0 \neq 0$ *for*

$$\mathbf{r}_\alpha = (\mathbf{r}_\alpha)_0, \qquad \tau_\beta = (\tau_\beta)_0, \qquad (\alpha = 1, \cdots, m, \beta = 1, \cdots, p).$$

The solution $\mathbf{G}$ *thus attained is unique, satisfies the condition*

$$(\mathbf{r}_i)_0 = \mathbf{G}[(\mathbf{r}_1)_0, \cdots, (\mathbf{r}_{i-1})_0, (\mathbf{r}_{i+1})_0, \cdots, (\mathbf{r}_m)_0, (\tau_1)_0, \cdots, (\tau_p)_0],$$

is continuous in the variables $\mathbf{r}_\alpha$, $(\alpha \neq i)$, τ_β, *and possesses continuous derivatives* $\nabla_{\mathbf{r}_\alpha}\mathbf{G}$, $(\alpha \neq i)$, $\partial\mathbf{G}/\partial\tau_\beta$.

Before proceeding with the proof we will first develop a mean value theorem for vector variables which will be useful in the sequel. Let $\mathbf{F}(\mathbf{r})$ be a vector function of the position vector $\mathbf{r}$ which is continuous and possesses a dyadic derivative $\nabla\mathbf{F}$ at all points of a region R; and let s measure the arc length along the vector $\mathbf{a}$ joining the current point $\mathbf{r}$ to the point $\mathbf{r} + \mathbf{a}$ and lying in R. Then, designating the modulus of $\mathbf{a}$ by α,

$$\left| \int_{\mathbf{a}}^{\mathbf{r}+\mathbf{a}} d\mathbf{F} \right| \leq \int_0^\alpha |\, d\mathbf{F}/ds \,|\, ds\,.$$

Applying the law of the mean to the right member,

$$(1) \qquad |\, \mathbf{F}(\mathbf{r} + \mathbf{a}) - \mathbf{F}(\mathbf{r}) \,| \leq \alpha\, |\, (d/ds)\mathbf{F}(\mathbf{r} + \theta\mathbf{a}) \,|\,, \qquad 0 \leq \theta \leq 1\,.$$

But

$$\alpha\, |\, (d/ds)\mathbf{F}(\mathbf{r} + \theta\mathbf{a}) \,| = |\, \mathbf{a} \cdot \nabla\mathbf{F}(\mathbf{r} + \theta\mathbf{a}) \,|\,.$$

Therefore

$$(2) \qquad \mathbf{F}(\mathbf{r} + \mathbf{a}) - \mathbf{F}(\mathbf{r}) = |\, \mathbf{a} \cdot \nabla\mathbf{F}(\mathbf{r} + \theta\mathbf{a}) \,|\, \mathbf{q}\,,$$

where $\mathbf{q}$ is a vector function of $\mathbf{r}$ whose modulus is less than or equal to unity. When $\mathbf{a}$ approaches zero, the vector $\mathbf{q}$ approaches a unit vector in the direction of the tangent at $\mathbf{F}(\mathbf{r})$ to the curve swept out by the terminus of $\mathbf{F}$ in the vector diagram. For, placing $\mathbf{a} = \alpha\mathbf{a}_1$

$$\lim_{\alpha \to 0} \mathbf{q} = \lim_{\alpha \to 0} \frac{[\mathbf{F}(\mathbf{r} + \mathbf{a}) - \mathbf{F}(\mathbf{r})]/\alpha}{| \mathbf{a}_1 \cdot \nabla \mathbf{F}(\mathbf{r} + \theta \mathbf{a}) |} = \frac{d\mathbf{F}/ds}{| d\mathbf{F}/ds |} = \frac{d\mathbf{F}}{d\sigma},$$

where σ is the arc length along any curve by which $\mathbf{F}(\mathbf{r} + \mathbf{a})$ approaches $\mathbf{F}(\mathbf{r})$.

For simplicity, the proof of the principal theorem will be carried out for $\mathbf{F}$ a function of two vector variables, $\mathbf{r}$ and $\mathbf{s}$, and a single scalar variable τ. The extension of the proof to functions of more variables will be obvious. Suppose, for definiteness, that we wish to solve for $\mathbf{r}$ and that $(\nabla_r \mathbf{F})_n \neq 0$ for all values of $\mathbf{r}$, $\mathbf{s}$, and τ which lie in a region

$$(3) \quad R\colon \alpha \equiv | \mathbf{r} - \mathbf{r}_0 | \leqq \lambda, \qquad \beta \equiv | \mathbf{s} - \mathbf{s}_0 | \leqq \lambda, \qquad | \tau - \tau_0 | \leqq \lambda.$$

Let $\mathbf{s}$ and τ take on any value satisfying (3) and let $\mathbf{r} = \mathbf{r}(s)$ be any curve of class C' in R such that

$$(4) \qquad \mathbf{t} \cdot \nabla_r | \mathbf{F} | \neq 0 \quad \text{for} \quad \nabla_r | \mathbf{F} | \neq 0,$$

where $\mathbf{t} = d\mathbf{r}/ds$. Such a curve, which we will call a *transverse* curve, intersects each of the level hyper-surfaces $| \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) | = \text{const.}$, and to each value of s there exists a unique level hyper-surface, $| \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) | = c$, where c is given by $| \mathbf{F}(\mathbf{r}(s), \mathbf{s}, \tau) | = c$. Since $(\nabla_r \mathbf{F})_n \neq 0$, it can be shown that along any transverse curve the vector function $\mathbf{F}$ is properly monotonic. If $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau)$ vanishes at a point $\mathbf{r}_1$, in R for any fixed values of $\mathbf{s}$ and τ other than $\mathbf{s}_0$ and τ_0 and satisfying (3), then there exists but one such point. For if a transverse curve be drawn between $\mathbf{r}_1$, and any other point for which $\mathbf{F}$ vanishes, then $| \mathbf{F} |$ would either remain equal to zero or reach a maximum on this curve and $(\nabla_r \mathbf{F})_n$ would vanish.

Since $\mathbf{F}$ is continuous the quantity λ can be so selected that

$$\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) \neq -k\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau),$$

where k is positive, for all values of $\mathbf{r}$, $\mathbf{s}$, and τ in R unless $\mathbf{F}$ vanishes in R.

Applying (2) to the identity

$$\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) = [\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) - \mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau)] + [\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau) - \mathbf{F}(\mathbf{r}_0, \mathbf{s}_0, \tau)] + [\mathbf{F}(\mathbf{r}_0, \mathbf{s}_0, \tau) - \mathbf{F}(\mathbf{r}_0, \mathbf{s}_0, \tau_0)],$$

we have

$$(5) \quad \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) = \mathbf{q}_1 | \mathbf{a} \cdot \nabla_r \mathbf{F}(\mathbf{r} + \theta_1 \mathbf{a}, \mathbf{s}, \tau) | + \mathbf{q}_2 | \mathbf{b} \cdot \nabla_s \mathbf{F}(\mathbf{r}_0, \mathbf{s} + \theta_2 \mathbf{b}, \tau) | + \mathbf{q}_3 | (\tau - \tau_0)(\partial/\partial\tau) \mathbf{F}[\mathbf{r}_0, \mathbf{s}_0, \tau + \theta_3(\tau - \tau_0)] | = 0,$$

where $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, are vector quantities whose moduli, k_1, k_2, k_3, are each less than or, equal to unity, and

$$\mathbf{a} = \mathbf{r} - \mathbf{r}_0 = \alpha \mathbf{a}_1 , \qquad \mathbf{b} = \mathbf{s} - \mathbf{s}_0 = \beta \mathbf{b}_1 ,$$

$\mathbf{a}_1$ and $\mathbf{b}_1$ being unit vectors in the direction of $\mathbf{a}$ and $\mathbf{b}$.

We can find positive numbers M and μ such that

$$k_1 \mid \mathbf{a}_1 \cdot \nabla_r \mathbf{F} \mid > M , \qquad k_2 \mid \mathbf{b}_1 \cdot \nabla_s \mathbf{F} \mid < \mu , \qquad k_3 \mid \partial \mathbf{F} / \partial \tau \mid < \mu ,$$

for all values of $\mathbf{r}$, $\mathbf{s}$, and τ satisfying (3). Now designate by ϵ a positive number less than λ and by η the smaller of λ and $M\epsilon/4\mu$, and assign to $\mathbf{s}$ and τ definite values which satisfy

$$(6) \qquad \mid \mathbf{s} - \mathbf{s}_0 \mid < \eta , \qquad \mid \tau - \tau_0 \mid < \eta .$$

We seek the roots of (5) wherein $\mathbf{F}$ is regarded as a function of $\mathbf{r}$, for $\alpha \leqq \epsilon$.

Evidently we can express $\mathbf{F}$ in the form

$$\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) = \mathbf{Q}(\mathbf{r}, \mathbf{s}, \tau) + \mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau) ,$$

where

$$\mathbf{Q}(\mathbf{r}, \mathbf{s}, \tau) = \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) - \mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau) = k_1 \alpha \mathbf{a}_1 \cdot \nabla_r \mathbf{F} ,$$

and where, owing to the manner in which η was selected

$$0 \leqq \mid \mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau) \mid < 2\eta\mu .$$

Since $\nabla_r \mathbf{Q} = \nabla_r \mathbf{F}$ we have $(\nabla_r \mathbf{Q})_n \neq 0$, and $\mid \mathbf{Q} \mid$ must increase from zero to a value greater than $4\eta\mu$ as α increases from zero to ϵ, irrespective of the direction of $\mathbf{a}_1$; i.e., the level hyper-surfaces $\mid \mathbf{Q} \mid$ = const. completely surround $\mathbf{r}_0$. It is possible to draw an arc γ, extending from $\mathbf{r}_0$ to the boundary of R or to a zero of $\mathbf{F}$ if one occurs for $\alpha \leqq \epsilon$, which is transverse to both sets of level surfaces, $\mid \mathbf{F} \mid$ = const. and $\mid \mathbf{Q} \mid$ = const., such that, proceeding along γ from $\mathbf{r}_0$, $\mid \mathbf{F} \mid$ will constantly decrease and $\mid \mathbf{Q} \mid$ will constantly increase. For, such an arc must satisfy only the conditions

$$(7) \qquad \mathbf{t} \cdot \nabla_r \mid \mathbf{F} \mid < 0 , \qquad \mathbf{t} \cdot \nabla_r \mid \mathbf{Q} \mid > 0, \qquad \mathbf{r}_0 = \mathbf{r}(s_0) ,$$

where the arc length s is so taken as to increase with $\mid \mathbf{Q} \mid$. Now $\nabla \mid \mathbf{P} \mid = \nabla \mathbf{P} \cdot \mathbf{P}_1$, where $\mathbf{P}$ is any vector and $\mathbf{P}_1$ is a unit vector in the direction of $\mathbf{P}$. By means of this relation the conditions (7) become equivalent to the conditions

$$\mathbf{t} \cdot \nabla_r \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) \cdot \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) < 0 , \qquad \mathbf{t} \cdot \nabla_r \mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) \cdot \mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau) < 0 ,$$

and the direction $\mathbf{t}$ of γ can always be chosen to satisfy these conditions since $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) \neq -k\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau)$. As $\mathbf{r}$ proceeds along γ in the direction of increasing s, $|\mathbf{Q}|$ must finally reach and then exceed $|\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau)|$ for $\alpha \leqq \epsilon$. Moreover, when $|\mathbf{Q}| = |\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau)|$, $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau)$ must vanish. For, let us suppose that $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau)$ does not vanish; then, since $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau)$ is monotonic, $|\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau)|$ must remain less than $|\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau)|$ in any interval on γ in which $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau)$ does not vanish, and the maximum value of $|\mathbf{Q}|$ for $\alpha \leqq \epsilon$ will be less than $2|\mathbf{F}(\mathbf{r}_0, \mathbf{s}, \tau)|$ which in turn is less than $4\eta\mu$.

It is therefore clear that the equation $\mathbf{F}(\mathbf{r}, \mathbf{s}, \tau) = 0$ may be solved (theoretically) for $\mathbf{r}$ in terms of $\mathbf{s}$ and τ,

$$\mathbf{r} = \mathbf{G}(\mathbf{s}, \tau),$$

in the neighborhood $|\mathbf{r} - \mathbf{r}_0| \leqq \epsilon$, $|\mathbf{s} - \mathbf{s}_0| < \eta$, $|\tau - \tau_0| < \eta$, of the values $\mathbf{r}_0$, $\mathbf{s}_0$, τ_0, and that $\mathbf{r}$ approaches $\mathbf{r}_0$ as $\mathbf{s}$ and τ approach $\mathbf{s}_0$ and τ_0, respectively. The function $\mathbf{G}$ is continuous for $\mathbf{s} = \mathbf{s}_0$, $\tau = \tau_0$ since $|\mathbf{r} - \mathbf{r}_0| < \lambda$, and likewise for any value $\mathbf{s}_1$, and τ_1, of $\mathbf{s}$ and τ satisfying the inequalities (6), since we have

$$|\mathbf{r} - \mathbf{r}_0| < \lambda, \qquad |\mathbf{s} - \mathbf{s}_0| < \lambda, \qquad |\tau - \tau_0| < \lambda.$$

Let τ be fixed at the value τ_0, then we have with the aid of (5) after multiplying by $\mathbf{a}_1$ and $\mathbf{q}_1$ and solving for $\mathbf{a}/\beta$

$$\frac{\mathbf{a}}{\beta} = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{s} - \mathbf{s}_0|} = -\frac{\mathbf{q}_1 \cdot \mathbf{q}_2 |\, \mathbf{b}_1 \cdot \nabla_{\mathbf{s}} \mathbf{F}(\mathbf{r}_0, \mathbf{s} + \theta_2 \mathbf{b}, \tau) \,|\, \mathbf{a}_1}{\mathbf{q}_1^2 |\, \mathbf{a}_1 \cdot \nabla_{\mathbf{r}} \mathbf{F}(\mathbf{r} + \theta_1 \mathbf{a}, \mathbf{s}, \tau) \,|}.$$

As $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{b}_1 \cdot \nabla_{\mathbf{s}} \mathbf{F}$, and $\mathbf{a}_1 \cdot \nabla_{\mathbf{r}} \mathbf{F}$ are continuous, and $(\nabla_{\mathbf{r}} \mathbf{F})_n$ and $\mathbf{q}_1$ do not vanish, the ratio $\mathbf{a}/\beta$ approaches a limit as β approaches zero, i.e., as $\mathbf{s}$ approaches $\mathbf{s}_0$; further, $\mathbf{r}$ approaches $\mathbf{r}_0$ as $\mathbf{s}$ approaches $\mathbf{s}_0$. In the limit

$$\frac{\partial \mathbf{r}}{\partial \sigma} = -\frac{\mathbf{T}_1 \cdot \mathbf{T}_2 |\, \bar{\mathbf{t}} \cdot \nabla_{\mathbf{s}} \mathbf{F} \,|\, \mathbf{t}}{|\, \mathbf{t} \cdot \nabla_{\mathbf{r}} \mathbf{F} \,|} = -\frac{\bar{\mathbf{t}} \cdot \nabla_{\mathbf{s}} \mathbf{F} \cdot \mathbf{T}_1 \mathbf{t}}{|\, \mathbf{t} \cdot \nabla_{\mathbf{r}} \mathbf{F} \,|} \tag{8}$$

where σ is the arc length along the curve by which $\mathbf{s}$ approaches $\mathbf{s}_0$, $\bar{\mathbf{t}}$ is the unit tangent to that curve for $\mathbf{s} = \mathbf{s}_0$, $\mathbf{t}$ is the corresponding unit tangent to the curve by which $\mathbf{r}$ approaches $\mathbf{r}_0$, and $\mathbf{T}_1$ and $\mathbf{T}_2$ are unit vectors in the directions of $\mathbf{t} \cdot \nabla_{\mathbf{r}} \mathbf{F}$ and of $\bar{\mathbf{t}} \cdot \nabla_{\mathbf{s}} \mathbf{F}$, respectively. The derivative $\partial \mathbf{r}/\partial \sigma$ thus exists and, as

$$\partial \mathbf{r}/\partial \sigma = \partial \mathbf{s}/\partial \sigma \cdot \nabla_{\mathbf{s}} \mathbf{r} = \bar{\mathbf{t}} \cdot \nabla_{\mathbf{s}} \mathbf{G},$$

the derivative $\nabla_{\mathbf{s}} \mathbf{G}$ also exists and, since the denominator of (8) cannot

vanish, it is continuous. Similarly, it may be shown that the derivative $\partial \mathbf{G}/\partial \tau$ exists and is continuous, and the theorem is proved. The extension of the proof to functions of several vector and scalar variables follows at once.

The general theorem for scalar functions of vector and scalar variables may be stated:

Any n scalar equations in m vector and p scalar variables

$$\phi_\alpha(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m, \tau_1, \tau_2, \cdots, \tau_p) = 0, \qquad (\alpha = 1, 2, \cdots, n), \tag{9}$$

may be solved for any vector $\mathbf{r}_i$ in terms of the remaining variables,

$$\mathbf{r}_i = \mathbf{G}_i(\mathbf{r}_1, \cdots, \mathbf{r}_{i-1}, \mathbf{r}_{i+1}, \cdots, \mathbf{r}_m, \tau_1, \cdots, \tau_p),$$

provided there exists values $(\mathbf{r}_\beta)_0$, $(\tau_\gamma)_0$, $(\beta = 1, 2, \cdots, m, \gamma = 1, 2, \cdots, p)$, which satisfy the above equations; that the functions ϕ_α are continuous and possess continuous partial derivatives $\nabla_{\mathbf{r}_\beta}\phi_\alpha$, $\partial\phi_\alpha/\partial\tau_\gamma$, in the neighborhood of $(\mathbf{r}_\beta)_0$, $(\tau_\gamma)_0$, and finally that the scalar product

$$[\nabla_{\mathbf{r}_i}\phi_1 \nabla_{\mathbf{r}_i}\phi_2 \cdots \nabla_{\mathbf{r}_i}\phi_n] \neq 0$$

for $\mathbf{r}_\beta = (\mathbf{r}_\beta)_0$, $\tau_\gamma = (\tau_\gamma)_0$. The solution so obtained is unique, satisfies the condition $(\mathbf{r}_i)_0 = \mathbf{G}_i[(\mathbf{r}_1)_0, \cdots, (\mathbf{r}_{i-1})_0, (\mathbf{r}_{i+1})_0, \cdots, (\mathbf{r}_m)_0, (\tau_1)_0, \cdots, (\tau_p)_0]$ is continuous in the variables $\mathbf{r}_\beta$, τ_γ, $(\beta \neq i)$, and possesses derivatives $\nabla_{\mathbf{r}_\beta}\mathbf{G}_i$, $\partial\mathbf{G}_i/\partial\tau_\gamma$, $(\beta \neq i)$, continuous for $\mathbf{r}_\beta = (\mathbf{r}_\beta)_0$, $\tau_\gamma = (\tau_\gamma)_0$.

This theorem follows immediately from the theorem on vector functions; for if we multiply the functions ϕ_α by the independent vectors $\mathbf{a}_\alpha$ respectively, we have

$$\sum_\alpha \phi_\alpha \mathbf{a}_\alpha = \mathbf{F}(\mathbf{r}_1, \cdots, \mathbf{r}_m, \tau_1, \cdots, \tau_p).$$

Differentiating with respect to $\mathbf{r}_i$ and then taking the nth of both members, we obtain

$$[\nabla_{\mathbf{r}_i}\phi_1 \nabla_{\mathbf{r}_i}\phi_2 \cdots \nabla_{\mathbf{r}_i}\phi_n][\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n] = (\nabla_{\mathbf{r}_i}\mathbf{F})_n.$$

$(\nabla_{\mathbf{r}_i}\mathbf{F})_n$ thus vanishes only with $[\nabla_{\mathbf{r}_i}\phi_1 \nabla_{\mathbf{r}_i}\phi_2 \cdots \nabla_{\mathbf{r}_i}\phi_n]$. Further, the derivatives $\nabla_{\mathbf{r}_\beta}\mathbf{F}$, $\partial\mathbf{F}/\partial\tau_\gamma$ will exist and be continuous in the neighborhood of the set of values $(\mathbf{r}_\beta)_0$, $(\tau_\gamma)_0$. The conditions of the theorem on vector functions are then satisfied and the theorem is proved.

The theorem concerning the solution of the set of scalar functions of scalar variables,

$$f_\alpha(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0, \qquad (\alpha = 1, 2, \cdots, n),$$

follows at once, for if we consider the x's as components of a vector $\mathbf{r}$ resolved along a set of mutually orthogonal axes, the system immediately becomes equivalent to the set (9), and the Jacobian $J\left(\frac{f_1, \cdots, f_n}{x_1, \cdots, x_n}\right)$ to the scalar product $[\nabla_r \phi_1 \nabla_r \phi_2 \cdots \nabla_r \phi_n]$. The resultant solution for $\mathbf{r}$ may then be resolved into the n scalar equations solved for the x's.

MINIMAL SURFACES OF GENERAL TOPOLOGICAL STRUCTURE WITH ANY FINITE NUMBER OF ASSIGNED BOUNDARIES[1]

By Jesse Douglas

1. **Introduction.** During the years 1926–1929 the author developed a complete solution of the problem of Plateau, the methods and results of which were presented to the American Mathematical Society at various meetings during those years, and published in the form of abstracts in the Bulletin of the American Mathematical Society. One of these abstracts was the following, entitled *A general formulation of the problem of Plateau.*[2] (Italics in the original text.)

"In this paper methods are developed for treating the problem of Plateau in the following general formulation. *Given k contours in n-dimensional euclidean space. To determine a minimal surface bounded by the k contours (and with no other boundaries) and having a prescribed topological structure, that is, genus and two-sidedness or one-sidedness. The minimal surface is, further, to be represented as a conformal image of a canonical configuration of the same topological structure.* The canonical configuration may always be represented as a region bounded by circles; we consider all possible ways of representing the contours as topological images of these circles:

$$x_i = g_i^{(1)}(\theta)\,, \qquad x_i = g_i^{(2)}(\theta), \cdots, \qquad x_i = g_i^{(k)}(\theta)\,.$$

The problem is then reduced to the minimization of a certain functional $A(g^{(1)}, g^{(2)}, \cdots, g^{(k)})$, generalizing $A(g)$ of the previous paper. The expression for this functional involves, in the place of the trigonometric functions occurring in $A(g)$, those functions which belong to a Riemann surface of the topological structure prescribed for the required minimal surface. For instance, for a minimal surface bounded by two given contours and to be represented conformally on a circular ring, the expression for $A(g^{(1)}, g^{(2)})$ involves elliptic functions, having periods 2π, $2p\sqrt{-1}$, where p denotes the logarithm of the ratio of the radii of

[1] Presented to the American Mathematical Society, April 27, 1936. Abstract in *Bull. Amer. Math. Soc.*, v. 42.

[2] Presented by title at the New York meeting, October, 1929. Published in *Bull. Amer. Math. Soc.*, v. 36 (1930), p. 50.

the circular ring. The existence of the required minimal surface depends on the existence of a real root for a certain definite equation in p. These ideas extend to the general case, where the moduli of Riemann intervene in place of the parameter p."

(The phrase "$A(g)$ of the previous paper" refers to the functional

$$A(g) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\sum_{i=1}^{n}[g(\theta)-g_i(\varphi)]^2}{4\sin^2\frac{\theta-\varphi}{2}}\,d\theta\,d\varphi\,,$$

by means of which the author obtained a complete solution of the Plateau problem for a simply-connected minimal surface with a given bounding curve.)

This was the first time that the Plateau problem, or, as far as the author knows, any calculus of variations problem, was stated with regard to topological generality of the extremal surface or for any finite number of contours.

In 1931, I published a paper entitled *The problem of Plateau for two contours*,[3] in which I gave the sufficient condition

$$m(\Gamma_1, \Gamma_2) < m(\Gamma_1) + m(\Gamma_2)$$

for the existence of a doubly-connected minimal surface bounded by two Jordan curves Γ_1, Γ_2, not mutually intersecting, and such that each is capable of bounding a surface of finite area: $m(\Gamma_1, \Gamma_2)$ is finite. $m(\Gamma_1, \Gamma_2)$ is the lower bound of the functional $A(g_1, g_2; q)$ or $A(g^{(1)}, g^{(2)})$ mentioned in the abstract; while $m(\Gamma_1)$, $m(\Gamma_2)$ are the lower bounds of $A(g)$ for Γ_1, Γ_2 respectively.

In case $m(\Gamma_1, \Gamma_2) = +\infty$, I established the sufficient condition

$$\delta(\Gamma_1, \Gamma_2) > 0\,,$$

where

$$\delta(\Gamma_1, \Gamma_2) = \limsup\{m(\Gamma_1') + m(\Gamma_2') - m(\Gamma_1', \Gamma_2')\}$$

as Γ_1', Γ_2' with $m(\Gamma_1', \Gamma_2')$ finite, approach Γ_1, Γ_2 in all the possible ways.

Although the title specified "Two contours," the formulas of that paper were developed for any finite number of contours[4] and the general topological structure of the minimal surface described in the abstract.

[3] This *Journal*, v. 10 (1931), pp. 315–359.

[4] Loc. cit., p. 324.

Exactly two things are necessary to extend the work of "Two contours" so as to afford a solution of the general case.

(1) The identity (loc. cit., p. 344)

$$w^2 P(z, w) P(\zeta, w) = - P(z, \zeta) \{z^2 Z_z(z, w) + \zeta^2 Z_\zeta(\zeta, w)\} + z\zeta\{Z(z, w) P_\zeta(z, \zeta) + Z(\zeta, w) P_z(z, \zeta)\} + \tfrac{1}{2} q \frac{\partial}{\partial q} P(z, \zeta)$$

must be generalized.

(2) The special variation (loc. cit., p. 344)

$$z' = z + \lambda z^2 Z(z, w)$$

must be generalized; and there will be the new difficulty, for genus >1 of the related Riemann surface, that the generalized variation must be proved a possible one.

The results of this generalization were recently published by the author in the paper *Some new results in the problem of Plateau.*[5] The present work goes on to give details and proofs. We shall refer freely to the paper just mentioned, under the designation "New results," and also to the paper "Two contours"—to such an extent that the reader should be well acquainted with these papers in order to follow the present one. We may also mention our paper *One-sided minimal surfaces with a given boundary,*[6] which belongs in the present series of publications, as well as the original paper *Solution of the problem of Plateau.*[7]

(In order to avoid many footnotes, we cite here the following two standing references. For algebraic geometry, Severi-Löffler, *Vorlesungen über algebraische Geometrie*, Leipzig and Berlin, 1921. For theta functions, H. F. Baker, *Abel's theorem and the allied theory including the theory of theta functions*, Cambridge University Press, 1897; especially chapter X.)

[5] Communicated to the National Academy of Sciences, July 1, 1935, for publication in its Proceedings; withdrawn July 12, 1935, because the length of the manuscript exceeded the limit of the equivalent of six printed pages; presented by title to the American Academy of Arts and Sciences, Boston, Mass., Nov. 13, 1935, see *Proc. Amer. Acad.*, v. 71; published in the preceding number of this *Journal*, v. 15, no. 1 (Feb., 1936), pp. 55–64.

[6] *Trans. Amer. Math. Soc.*, v. 34 (1932), pp. 731–756.

[7] *Trans. Amer. Math. Soc.*, v. 33 (1931), pp. 263–321.

2. **General bounding curves.** Before proceeding to our main purpose, we state here a fourth theorem, supplementing the three given in "New results." This theorem refers to boundaries $\bar{\Gamma}_1, \bar{\Gamma}_2, \cdots, \bar{\Gamma}_k$ in the form of completely general Jordan curves—where, then, usually:

$$m_r(\bar{\Gamma}_1, \bar{\Gamma}_2, \cdots, \bar{\Gamma}_k) = +\infty,$$

which implies that the given curves cannot bound any surface of finite area.

THEOREM IV. *A sufficient condition that k given Jordan curves $\bar{\Gamma}_1 \bar{\Gamma}_2, \cdots, \bar{\Gamma}_k$ not mutually intersecting, be the boundaries of a two-sided minimal surface with (necessarily even) Betti number r, is that it be possible to approach to $\bar{\Gamma}_1, \bar{\Gamma}_2, \cdots, \bar{\Gamma}_k$ with a sequence of contour-systems $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ such that $m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k)$ is always finite, and where the system of (essentially non-negative) quantities*

$$m_{r'}(\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_i) + m_{r''}(\Gamma''_1, \Gamma''_2, \cdots, \Gamma''_j) - m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

$$m_{r-2}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) - m_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

formed for all partitions:

$$(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) = (\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_i) + (\Gamma''_1, \Gamma''_2, \cdots, \Gamma''_j),$$

$$r = r' + r'' \qquad (r', r'' \text{ even}),$$

shall remain $> \epsilon > 0$.

This is to say that the superior limit of the least of the above quantities shall be positive.

A similar statement applies to *one-sided minimal surfaces*, with the preceding system of quantities replaced by the following:

$$\bar{m}_{r'}(\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_i) + \bar{m}_{r''}(\Gamma''_1, \Gamma''_2, \cdots, \Gamma''_j) - \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

$$\bar{m}_{r'}(\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_i) + m_{r''}(\Gamma''_1, \Gamma''_2, \cdots, \Gamma''_j) - \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

$$\bar{m}_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) - \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k),$$

$$m_{r-1}(\Gamma_1, \Gamma_2, \cdots, \Gamma_k) - \bar{m}_r(\Gamma_1, \Gamma_2, \cdots, \Gamma_k).$$

The second and fourth are employed, respectively, only for r'' even and r odd.

We reserve the proof of theorem IV for another publication.

3. **Formulas.** The Green's function relative to the point w of half a symmetric Riemann surface $\mathfrak{R}$, of any finite genus p, has the form

$$G(z, w) = \Re \log \frac{S(z, w)}{S_1(z, \overline{w})}, \tag{1}$$

where $\overline{w}$ denotes the conjugate point to w. Explicitly, we have

$$G(z, w) = \Re \log \frac{\theta(v(z, w); \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha; \tau)}{\theta(v(z, \overline{w}); \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha; \tau)}, \tag{2}$$

where θ denotes the function of u_j, τ_{jk} ($j, k = 1, 2, \cdots, p$) defined by the infinite series

$$\begin{aligned}\theta(u; \tfrac{1}{2}\beta, \tfrac{1}{2}\alpha; \tau) = \Sigma \exp\,[&2\pi i\, u_j(n_j + \tfrac{1}{2}\alpha_j) \\ &+ i\pi\, \tau_{jk}(n_j + \tfrac{1}{2}\alpha_j)\,(n_k + \tfrac{1}{2}\alpha_k) + i\pi\, \beta_j(n_j + \tfrac{1}{2}\alpha_j)]\,.\end{aligned} \tag{3}$$

The summation convention is employed for the indices j, k, and Σ denotes a multiple summation as $n_1, n_2, \cdots, n_p$ range over all integers from $-\infty$ to $+\infty$.

$\frac{1}{2}\beta, \frac{1}{2}\alpha$, called the *characteristic* of the θ-function, is supposed to consist of $2p$ half-integers $\frac{1}{2}\beta_j, \frac{1}{2}\alpha_j$, usually 0 or $\frac{1}{2}$. We require

$$\beta\alpha = \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_p\alpha_p \tag{3'}$$

to be odd (*odd characteristic*).

$v(z, w)$ denotes the system of p normal integrals of the first kind, from the lower limit w to the upper limit z: $v_j(z, w) = v_j(z) - v_j(w)$.

Formula (2) recites that $G(z, w)$ is the real part of the normal integral of the third kind for the symmetric Riemann surface, with logarithmic singularities at w and $\overline{w}$.

This gives by "Two contours"[8] the following formula for the functional $A(g; \mu)$, where μ denotes the $3p - 3$ moduli of Riemann, or $A(g; R)$,

$$A(g; R) = \frac{1}{4\pi} \int_c \int_c \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2 \frac{\partial^2}{\partial z\, \partial \zeta} \log \theta(v(z, \zeta); \tau)\, dz\, d\zeta\,. \tag{4}$$

z, ζ vary independently over the boundary $C = C_1 + C_2 + \cdots + C_k$ of R', one of the symmetric halves of R. Although complex quantities occur in the expression (4), the value of $A(g; R)$ is always positive real—for one reason, on account of the symmetry of the Riemann surface—for another, because $A(g; R)$ is equal to half the sum of the Dirichlet integrals of the harmonic functions on R' which have the values $g_i(z)$ on its boundary C.

[8] Loc. cit., formulas (2.18), (2.15).

An equivalent expression is*

$$A(g; R) = \frac{1}{4\pi} \int_c \int_c \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2 \tag{5}$$
$$\wp_{jk}(v(z, \zeta); \tau)\, v_j'(z)\, v_k'(\zeta)\, dz d\zeta ,$$

where $\wp_{jk}$ is a generalization of the elliptic function $\wp$ of Weierstrass:

$$\wp_{jk}(u; \tau) = -\frac{\partial^2 \log \theta(u; \tau)}{\partial u_j\, \partial u_k} . \tag{6}$$

We shall also need the generalization of the elliptic function ζ:

$$\zeta_j(u; \tau) = \frac{\partial \log \theta(u; \tau)}{\partial u_j} . \tag{7}$$

The harmonic functions

$$x_i = \Re F_i(w) \qquad (i = 1, 2, \cdots, n) , \tag{8}$$

determined on the semi Riemann surface R' by the boundary values on C:

$$x_i = g_i(z) , \tag{9}$$

are expressed by the formula

$$F_i(w) = \frac{1}{\pi i} \int_c g_i(z) \frac{\partial}{\partial z} \log \theta(v(z, w); \tau)\, dz + c_i , \tag{10}$$

where the c_i are constants to be properly determined. An equivalent form of (10) is

$$F_i(w) = \frac{1}{\pi i} \int_c g_i(z)\, \zeta_j(v(z, w); \tau) v_j'(z)\, dz + c_i . \tag{11}$$

By permissible differentiation under the integral sign, we get

$$F_i'(w) = \frac{1}{\pi i} \int_c g_i(z)\, \wp_{jl}(v(z, w); \tau)\, v_j'(z)\, v_l'(w)\, dz . \tag{12}$$

From this, we obtain

$$\sum_{i=1}^{n} F_i'^2(w) = \frac{1}{2\pi^2} \int_c \int_c \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2 \tag{13}$$
$$\wp_{jl}(v(z, w); \tau)\, \wp_{km}(v(\zeta, w); \tau)\, v_j'(z)\, v_k'(\zeta)\, v_l'(w)\, v_m'(w)\, dz\, d\zeta .$$

* Throughout this paper, the summation convention as to repeated indices will be used for j, k, l, m, which range from 1 to p. The range of the index i is

That the minimum of $A(g; R)$ is attained for a proper representation

$$x_i = g_{\alpha i}(z) \qquad (\alpha = 1, 2, \cdots, k) \tag{14}$$

of Γ (locus of (x_i)) on C (locus of z), and for a non-degenerate Riemann surface R, follows, as many times before in the author's previous papers, by the theory of lower semi-continuous functionals on compact closed sets. The minimum is in comparison with all symmetric Riemann surfaces R, one of the conjugate halves of which, R', is topologically equivalent to the required minimal surface; and for all topological representations of the ensemble of given contours

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_k$$

on the ensemble of boundary curves of R': $C = C_1 + C_2 + \cdots + C_k$. That improper or degenerate forms of topological representation cannot produce a minimum can be shown as in "Two contours," and the possibility of degeneration of the Riemann surface R is avoided by the inequalities postulated in the statements of theorems I and II of "New results"—that is the purpose of those inequalities.

It is the vanishing of the first variation of $A(g; R)$:

$$\delta A(g; R) = 0$$

for the minimizing g and R, with regard to any variation of the parametric representation g of Γ, or of the Riemann surface R, that will give us the condition

$$\sum_{i=1}^{n} F_i'^2(w) = 0\,,$$

characteristic of a minimal surface.

4. **Variation of the representation** $(g; R)$. As in our previous papers, we use a properly arranged special variation of the Riemann surface R and the representation g of Γ. The one that will serve our purpose is

$$\delta v_j(z) = -\lambda v_j'(w)t(z, w) = -\lambda v_j'(w)v_k'(w)\zeta_k(v(z, w); \tau)\,, \tag{15}$$

where λ is a parameter sufficiently small in absolute value. $t(z, w)$ is Riemann's notation for the normal integral of the second kind with pole at w.

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from 1 to n, the number of dimensions of the euclidean space which contains the given contours. Besides i the index, $i = \sqrt{-1}$ will occur frequently, without possibility of confusion.

Since the periods of $t(z, w)$ are zero with respect to the first system of circuits, and $-2\pi i v_k'(w)$ with respect to the second system, we have for the variation of the periods of the integrals of the first kind:

$$\delta\tau_{jk} = 2\pi i \lambda v_j'(w) v_k'(w). \tag{16}$$

Now, the variation (15) is, itself, not permissible at all, since it does not preserve the curves of transition C of the Riemann surface R (or the real branches of the corresponding algebraic curve). But it can be expressed as a linear combination of variations which preserve the symmetry of R, together with its curves of transition—and the latter, if λ is small enough absolutely, in a one-one continuous way; a condition entirely necessary, since the minimum of $A(g; R)$ is with respect to *topological* representations of Γ on the *curves of transition* of a *symmetric* Riemann surface.

These permissible variations are

$$\delta v_j(z) = \lambda \frac{v_j'(w)t(z, w) + v_j'(\overline{w})t(z, \overline{w})}{2}, \tag{17'}$$

$$\delta v_j(z) = \lambda \frac{v_j'(w)t(z, w) - v_j'(\overline{w})t(z, \overline{w})}{2i}, \tag{17''}$$

where λ is a *real* parameter with $|\lambda| < \epsilon$, a properly chosen positive number. On the curves of transition, $z = \bar{z}$, so that the fractions represent respectively the sum divided by 2, and the difference divided by $2i$, of two conjugate complex quantities; these fractions are therefore real quantities. Consequently, the variations (17'), (17'') preserve the reality of $v_j(z)$; accordingly they keep the curves of transition. They do this in a one-one continuous way if λ is sufficiently small in absolute value. For if $\lambda = 0$, these variations reduce to the identity, where the rate of motion of the transformed point with respect to the original one is unity; now, for λ near enough to zero, this rate stays close enough to unity to be positive—which means that we have a monotonic continuous, or a one-one continuous, transformation.

The *formal* variation (15) is a linear combination with coefficients 1, i of the *actual* variations (17'), (17''). Hence the formal variation produced in $A(g; R)$ by (15) must be the same linear combination of the actual variations produced by (17'), (17''):

$$\delta A(g; R) = \delta' A(g; R) + i\delta'' A(g; R), \tag{18}$$

δ referring to (15), δ' to (17'), δ'' to (17'').

Well, the point is, as we shall prove at some length, that

$$\delta A(g; R) = \lambda \cdot \frac{\pi}{2} \sum_{i=1}^{n} F_i'^2(w) \; ; \tag{19}$$

so that $\delta' A(g; R)$ and $\delta'' A(g; R)$ are, except for an inessential factor, the real and imaginary components respectively of $\sum_{i=1}^{n} F_i'^2(w)$—and since, as actual variations,

$$\delta' A(g; R) = 0 \, , \qquad \delta'' A(g; R) = 0 \, , \tag{20}$$

we shall have

$$\sum_{i=1}^{n} F_i'^2(w) = 0 \, , \tag{21}$$

the condition for a minimal surface.

Conversely, if the variation (15) of the Riemann surface R can be proved possible, the variations (17′), (17″) will be possible, as linear combinations of (15) formed for w and its conjugate $\bar{w}$.

For $p > 1$, a serious difficulty is to prove the possibility of the simultaneous variation (15) of the p normal integrals of the first kind. Essentially equivalent to this is the possibility of varying the periods τ_{jk} according to (16). The number of these periods is $\frac{1}{2} p(p+1)$, and there only $3p - 3$ arbitrary moduli at our disposal; now, if $p > 3$, then

$$\tfrac{1}{2} p(p+1) > 3p - 3 \, . \tag{22}$$

This being the case, proof is required that the variations (15) and (16) are possible by proper variation of the Riemann surface.

We shall prove the possibility of (15), from which that of (16) follows immediately by a circuit of the second system.

We employ the Clebsch-Lüroth canonical form of the Riemann surface. Let w denote any branch-point, which will be varied, while the other branch-points are kept fixed. Our problem is to determine the effect on $v_j(z)$ of this variation of a single branch-point.

We proceed in an entirely qualitative way. The characteristic property of the integrals of the first kind is that they can be prolonged indefinitely without coming to any singularity; that is, any branch of $v_j(z)$ can be expressed in the vicinity of any place on the Riemann surface as a convergent power series in the local uniformizing variable. This will be $t = z - a$ for a finite place a not a branch-point, $t = \sqrt{z - w}$ for the branch-point w, $t = \sqrt{z - w'}$ for a branch-point $w' \neq w$, and $t = \frac{1}{z}$ for $z = \infty$.

Thus, about the place $z = a$, we have

$$v_j(z) = v_j(0) + v_j'(0)t + \cdots, \tag{23}$$

the argument 0 referring to $t = 0$. Consequently,

$$\frac{\partial v_j(z)}{\partial w} = \frac{\partial v_j(0)}{\partial w} + \frac{\partial v_j'(0)}{\partial w}\, t + \cdots \tag{24}$$

(t does not depend on w)—which is regular about the place a.

About the branch-point $w' = w$, or about $z = \infty$, the same applies, for t does not depend on w. But about the branch-point w itself—the one that is being varied—we have $t = \sqrt{z - w}$ so that

$$\frac{\partial t}{\partial w} = -\frac{1}{2\sqrt{z - w}} = -\frac{1}{2t}.$$

Therefore, about this place,

$$\begin{aligned}\frac{\partial v_j(z)}{\partial w} = &-\tfrac{1}{2} v_j'(0) \cdot \frac{1}{t} + \left\{\frac{\partial v_j(0)}{\partial w} - \tfrac{1}{2} v_j''(0)\right\} \\ &+ \left\{\frac{\partial v_j'(0)}{\partial w} - \tfrac{1}{2} v_j'''(0)\right\} t + \cdots;\end{aligned} \tag{25}$$

that is, at the branch-point w there is a pole of the first order with residue $-\frac{1}{2} v_j'(w)$, where the differentiation is with respect to the local uniformizing variable.

Hence $\partial v_j(z)/\partial w$ is a function prolongable indefinitely on the Riemann surface, with its only singularity a pole of the first order at w. Therefore $\partial v_j(z)/\partial w$ is an integral of the second kind. Its periods with respect to the first system of circuits are all zero, since those of $v_j(z)$ are constantly equal to the Kronecker symbols $\delta_{j\lambda}$ during the variation of w. This makes $\partial v_j(z)/\partial w$ proportional to the *normal* integral of the second kind with pole at w, denoted by $t(z, w)$; and on account of the above value of the residue of w, we have exactly:

$$\frac{\partial v_j(z)}{\partial w} = -\tfrac{1}{2} v_j'(w)\, t(z, w). \tag{26}$$

Thus the possibility of the variation (15) is proved in case w is a branch-point. The value of z is kept fixed; that is, the varied curve is set into correspondence with the fixed one, essentially, by intersection with straight lines parallel to the tangent at w. More generally—and for the vicinity of w, more exactly—the correspondence is by inter-

section with a family of curves, one through each point P of the initial curve, and whose tangent at P is parallel to the tangent at w of the initial curve. For instance, a family of parabolas may be used, whose latus rectum varies regularly with P and reduces to zero when P coincides with w.

A branch-point of an algebraic curve is merely one where the tangent line is parallel to the y-axis; and the tangent at any point w of the curve can be made parallel to the y-axis by rotation of axes, a birational transformation, which changes nothing essential. Briefly, a branch-point in algebraic geometry is not intrinsically different from any other point on the curve—so that the variation (15) is thus proved possible for a general point w on the Riemann surface. The transformation that will produce (15) is, explicitly, TVT^{-1}, where T is the transformation of axes that makes w a branch-point, and V is the displacement of this branch-point.

5. **Variation of $A(g; R)$.** We have to perform the variations (15), (16) on the expression (5) for $A(g; R)$, which we repeat here:

$$A(g;R) = \frac{1}{4\pi}\int_e\int_e \sum_{i,j=1}^{n} [g_j(z) - g_i(\zeta)]^2\, \varphi_{jk}(v(z,\zeta))\, dv_j(z)\, dv_k(\zeta)\,, \tag{27}$$

where to abbreviate, we shall hereafter leave implicit the dependence on τ of θ, ζ and φ.

On the elements of this integral the effect of the variation is as follows:

$$\begin{aligned}\delta\varphi_{jk}(v(z,\zeta)) &= \varphi_{jkl}(v(z,\zeta))\,\{\delta v_l(z) - \delta v_l(\zeta)\} + \frac{\partial\varphi_{jk}(v(z,\zeta))}{\partial\tau_{lm}}\,\delta\tau_{lm}\\ &= \lambda\Big\{-\varphi_{jkl}(v(z,\zeta))\,v_l'(w)\,v_m'(w)\,\zeta_m(v(z,w))\\ &\quad + \varphi_{jkl}(v(z,\zeta))\,v_l'(w)\,v_m'(w)\,\zeta_m(v(\zeta,w))\\ &\quad + 2\pi i\,v_l'(w)\,v_m'(w)\,\frac{\partial\varphi_{jk}(v(z,\zeta))}{\partial\tau_{lm}}\Big\}\,,\end{aligned} \tag{28}$$

where

$$\varphi_{jkl}(u;\tau) = \frac{\partial\varphi_{jk}(u;\tau)}{\partial u_l}\,; \tag{28'}$$

evidently φ_{jkl} is symmetric in j, k, l.

$$\delta dv_j(z) = d\delta v_j(z) = \lambda v_j'(w)v_l'(w)\,\varphi_{lm}(v(z,w))\,v_m'(z)\,dz\,. \tag{29}$$

$$\delta dv_k(\zeta) = d\delta v_k(\zeta) = \lambda v_k'(w)v_l'(w)\,\varphi_{lm}(v(\zeta,w))\,v_m'(\zeta)\,d\zeta\,. \tag{30}$$

By the product rule of differentiation:

$$\begin{aligned}(31)\qquad &\delta[\wp_{jk}(v(z,\zeta))\,dv_j(z)\,dv_k(\zeta)]\\ &= dv_j(z)\,dv_k(\zeta)\,\delta\wp_{jk}(v(z,\zeta)) + \wp_{jk}(v(z,\zeta))\,dv_k(\zeta)\,\delta\,dv_j(z)\\ &\qquad + \wp_{jk}(v(z,\zeta))\,dv_j(z)\,\delta dv_k(\zeta)\,.\end{aligned}$$

From this, by application of (28), (29), (30) with change of summation indices to simplify the resulting expression, we get

$$\begin{aligned}(31')\qquad &\delta[\wp_{jk}(v(z,\zeta))\,dv_j(z)\,dv_k(\zeta)]\\ &= \lambda\Big\{\wp_{jkl}(v(z,\zeta))\,[-\zeta_m(v(z,w)) + \zeta_m(v(\zeta,w))]\\ &\quad + \wp_{kl}(v(z,\zeta))\,\wp_{jm}(v(z,w)) + \wp_{jm}(v(z,\zeta))\,\wp_{kl}(v(\zeta,w))\\ &\quad + 2\pi i\,\frac{\partial\wp_{jk}(v(z,\zeta))}{\partial\tau_{lm}}\Big\}\,v_j'(z)\,v_k'(\zeta)\,v_l'(w)\,v_m'(w)\,dz\,d\zeta\,.\end{aligned}$$

Referring back to (27), we therefore have the formula

$$\begin{aligned}(32)\qquad &\delta A(g;R) = \lambda\cdot\frac{1}{4\pi}\int_e\int_e\sum_{i=1}^{n}[g_i(z) - g_i(\zeta)]^2\\ &\Big\{\wp_{jkl}(v(z,\zeta))\,[-\zeta_m(v(z,w)) + \zeta_m(v(\zeta,w))]\\ &\quad + \wp_{kl}(v(z,\zeta))\,\wp_{jm}(v(z,w)) + \wp_{jm}(v(z,\zeta))\,\wp_{kl}(v(\zeta,w))\\ &\quad + 2\pi i\,\frac{\partial\wp_{jk}(v(z,\zeta))}{\partial\tau_{lm}}\Big\}\,v_j'(z)\,v_k'(\zeta)\,v_l'(w)\,v_m'(w)\,dz\,d\zeta\,.\end{aligned}$$

6. **The identity.** Well, this expression simplifies remarkably, for *we have the identity*

$$\begin{aligned}(33)\qquad &\Big\{\wp_{jkl}(v(z,\zeta))\,[-\zeta_m(v(z,w)) + \zeta_m(v(\zeta,w))]\\ &\quad + \wp_{kl}(v(z,\zeta))\,\wp_{jm}(v(z,w)) + \wp_{jm}(v(z,\zeta))\,\wp_{kl}(v(\zeta,w))\\ &\quad + 2\pi i\,\frac{\partial\wp_{jk}(v(z,\zeta))}{\partial\tau_{lm}}\Big\}\,v_j'(z)\,v_k'(\zeta)\,v_l'(w)\,v_m'(w)\\ &\qquad = \wp_{jl}(v(z,w))\,\wp_{km}(v(\zeta,w))\,v_j'(z)\,v_k'(\zeta)\,v_l'(w)\,v_m'(w),\end{aligned}$$

which is proved in the next section.

We consider this identity—which, as far as we know, is new—to be one of the most important by-products of our work. It subsists for

any Riemann surface whatever of finite genus (algebraic curve), and has interest and value quite apart from any connection with the Plateau problem, of which it is entirely independent.

Assuming then the formula (33), in anticipation of its proof in §7, and applying it to (32), we get, finally:

$$\delta A(g; R) = \delta' A(g; R) + i\,\delta'' A(g; R)$$

$$= \lambda \cdot \frac{1}{4\pi} \int_e \int_e \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2\, \varphi_{jl}(v(z, w))\, \varphi_{km}(v(\zeta, w)) \tag{34}$$

$$v_j'(z)\, v_k'(\zeta)\, v_l'(w)\, v_m'(w)\, dz\, d\zeta\,.$$

But, according to (13), this is to say that

$$\delta A(g; R) = \delta' A(g; R) + i\,\delta'' A(g; R) = \lambda \cdot \frac{\pi}{2} \sum_{i=1}^{n} F_i'^2(w)\,. \tag{35}$$

Consequently, on account of the vanishing of $\delta' A(g; R)$ and $\delta'' A(g; R)$, we have

$$\sum_{i=1}^{n} F_i'^2(w) = 0\,. \tag{36}$$

In sum, if the system $g; R$ minimizes $A(g; R)$, then the surface

$$x_i = \Re\, F_i(w) \tag{37}$$

determined by the formula (11) is minimal. w ranges over one of the symmetric halves, R', of R, so that this minimal surface has the topological structure of R'. It is bounded by the given contours in their parametric representation

$$x_i = g_i(z)\,, \tag{38}$$

since these are the values on the boundaries of R' of the harmonic functions (37).

Remarks on the identity (33). Our task in connection with the identity (33) was this. We had the second member of this identity as the kernel of the expression (13) for $\sum_{i=1}^{n} F_i'^2(w)$ as a double integral. We had (1) to discover the first member; (2) prove its equality to the second; (3) then discover a variation of $v_j(z)$ and τ_{jk} that would produce this first member as the kernel of the integral expression for $\delta A(g; R)$; (4) then prove, by giving an explicit construction of this variation, that its real and imaginary parts could actually be performed on a sym-

metric Riemann surface, accompanied by a topological transformation of the curves of transition—that is, on a real algebraic curve, accompanied by topological transformation of the real branches.

As a guide, we had the identity of "Two contours" reproduced toward the beginning of this paper, but this was much less help than might be supposed. Broadly speaking, it was like generalizing a geometric result from only one dimension to n dimensions—and, of course, the problem first had to be looked at in this way. In "Two contours," the basic Riemann surface consists of the two faces of a circular ring, or is a torus, with genus = 1; so that the two-contour case is really the case 1 of the Plateau problem, the one-contour problem being the case 0.

The identity of "Two contours" is one in elliptic functions, and can be traced back to

$$[\zeta(u+v)-\zeta(u)-\zeta(v)]^2=\wp(u+v)+\wp(u)+\wp(v)\,, \tag{39}$$

together with the partial differential equation

$$\frac{\partial^2\theta}{\partial u^2}=4\pi i\frac{\partial\theta}{\partial\tau} \tag{40}$$

obeyed by the elliptic θ-functions of Jacobi.

The identity (39) is on the usual definition of ζ and $\wp$ in terms of σ. If instead we use a ζ and $\wp$ derived from θ_1:

$$\zeta(u)=\frac{d\log\theta_1(u)}{du}\,,\qquad \wp(u)=-\frac{d^2\log\theta_1(u)}{du^2}\,,$$

then the form of the identity (39) is

$$[\zeta(u+v)-\zeta(u)-\zeta(v)]^2=\wp(u+v)+\wp(u)+\wp(v)+\text{constant}\,, \tag{39'}$$

where the constant depends on the periods of the elliptic functions.

The identity of "Two contours" can be obtained by applying $\partial^2/\partial u\partial v$ to (39′), and then using (40) to simplify the result. The constant term in (39′) is thus without final influence.

7. **Proof of the identity.** For the case of a general Riemann surface ($p>1$), we shall now establish in place of (39′) the identity:

$$\begin{aligned}&[\zeta_l(v(z,w))-\zeta_l(v(z,\zeta))-\zeta_l(v(\zeta,w))]\,[\zeta_m(v(z,w))-\zeta_m(v(z,\zeta))\\&\qquad-\zeta_m(v(\zeta,w))]\,v_l'(w)\,v_m'(w)\\&=[\wp_{lm}(v(z,w))+\wp_{lm}(v(z,\zeta))+\wp_{lm}(v(\zeta,w))]\\&\qquad v_l'(w)\,v_m'(w)+\text{function }(w)\end{aligned} \tag{41}$$

(observe that $v(z,w)=v(z,\zeta)+v(\zeta,w)$).

In place of (40) we have the relation

$$\frac{\partial^2\theta(u;\tau)}{\partial u_l\,\partial u_m} = 4\pi i\,\frac{\partial\theta(u;\tau)}{\partial\tau_{lm}}, \tag{42}$$

to be seen immediately from the series (3) for θ.

The proof of (41) will be by the theorem of Riemann-Roch.

Consider each side of (41) as a function of z alone, ζ and w being regarded as parameters.

Then $\wp_{lm}(v(z,w))$, $\wp_{lm}(v(z,\zeta))$ are uniform functions of z, while both $\zeta_l(v(z,w))$ and $\zeta_l(v(z,\zeta))$ have the period 0 with respect to all the circuits of the first system, and the period $-2\pi i\delta_{lm}$ with respect to the m'th circuit of the second system. It follows that each bracket on the left of (41) is a uniform function on the Riemann surface of z.

Hence the difference between the first and second members of (41) (omitting the function (w)) is a uniform function on the Riemann surface; and has at most polar singularities, since these are the only ones manifested by $\wp_{lm}$ or ζ_l, ζ_m. Consequently, this difference is a *rational function*; and we proceed to prove that it is, indeed, a *constant*.

Using the subscripts l, m to denote partial differentiation of $\theta(u)$ with respect to u_l, u_m, we have, by definition:

$$\zeta_l = \frac{\theta_l}{\theta}, \qquad \wp_{lm} = -\frac{\partial\zeta_l}{\partial u_m} = \frac{\theta_l\,\theta_m}{\theta^2} - \frac{\theta_{lm}}{\theta} = \zeta_l\,\zeta_m - \frac{\theta_{lm}}{\theta}. \tag{42'}$$

Accordingly,

$$\begin{aligned}
&[\zeta_l(v(z,w)) - \zeta_l(v(z,\zeta)) - \zeta_l(v(\zeta,w))]\,[\zeta_m(v(z,w)) - \zeta_m(v(z,\zeta)) \\
&\qquad - \zeta_m(v(\zeta,w))]\,v_l'(w)\,v_m'(w) \\
&- [\wp_{lm}(v(z,w)) + \wp_{lm}(v(z,\zeta)) + \wp_{lm}(v(\zeta,w))]\,v_l'(w)\,v_m'(w) \\
&= \left\{\frac{\theta_{lm}(v(z,w))}{\theta(v(z,w))} + \frac{\theta_{lm}(v(z,\zeta))}{\theta(v(z,\zeta))} + \frac{\theta_{lm}(v(\zeta,w))}{\theta(v(\zeta,w))}\right. \\
&\quad - 2\,\frac{\theta_l(v(z,w))\,\theta_m(v(z,\zeta))}{\theta(v(z,w))\,\theta(v(z,\zeta))} - 2\,\frac{\theta_l(v(z,w))\,\theta_m(v(\zeta,w))}{\theta(v(z,w))\,\theta(v(\zeta,w))} \\
&\quad \left. + 2\,\frac{\theta_l(v(z,\zeta))\,\theta_m(v(\zeta,w))}{\theta(v(z,\zeta))\,\theta(v(\zeta,w))}\right\} v_l'(w)\,v_m'(w).
\end{aligned} \tag{43}$$

The function $\theta(v(z,\zeta))$ vanishes to the first order at $z=\zeta$, and at

exactly $p-1$ other points $a_1, a_2, \cdots, a_{p-1}$, *which are independent*[9] *of* ζ—so that the function $\theta(v(z, w))$ vanishes to the first order at $z = w$ and the *same* $p-1$ other points.

The right member of (43) is regular, except perhaps where $\theta(v(z, \zeta)) = 0$ or $\theta(v(z, w)) = 0$—there, it may have a pole. We shall prove that, in fact, $z = \zeta$ is not a pole; and, by similar reasoning, neither is $z = w$.

Proof that $z = \zeta$ *is not a pole.* Evidently, it is at most a pole of first order, and its residue is found by putting $z = \zeta$ in the numerator and in the derivative of the denominator of each fraction in (43) whose denominator vanishes at $z = \zeta$. This gives for the residue:

$$\frac{\theta_{lm}(0)\, v_l'(w)\, v_m'(w)}{\theta_k(0)\, v_k'(\zeta)} - 2\,\zeta_l(v(\zeta, w))\,\frac{\theta_m(0)}{\theta_k(0)\, v_k'(\zeta)}\, v_l'(w)\, v_m'(w) + 2\,\zeta_m(v(\zeta, w))\,\frac{\theta_l(0)}{\theta_k(0)\, v_k'(\zeta)}\, v_l'(w)\, v_m'(w)\,. \tag{44}$$

The last two terms mutually cancel, differing only by the interchange of the summation indices l, m. This leaves only the first term, which is zero, because the θ-function with odd characteristic[10] is an odd function of the variables u_l; hence its Maclaurin series contains only odd powers of u; consequently, all partial derivatives of even order vanish at $u = 0$; in particular, $\theta_{lm}(0) = 0$ for all l, m.

In sum, $z = \zeta$ is a pole of the first order with residue equal to zero; that is, it is not a pole at all.

The same argument applies, as stated, to $z = w$.

It remains to examine the places $a_1, a_2, \cdots, a_{p-1}$. A look at the term

$$-2\,\frac{\theta_l(v(z, w))\,\theta_m(v(z, \zeta))}{\theta(v(z, w))\,\theta(v(z, \zeta))}\, v_l'(w)\, v_m'(w) \tag{45}$$

in (43) suggests that these places might be poles of the second order; but we now show that, in fact, they are at most poles of the first order.

For, as remarked above, the function $\theta(v(z, w))$ vanishes for $z = a_1$ *regardless of the value of* w (to repeat: the zeros of this function other than w are independent[9] of w). That is, the equation

$$\theta(v(a_1, w)) = 0 \tag{46}$$

[9] Baker, loc. cit., p. 262, (IV).

[10] See formula (3′).

is an identity in w. It may therefore be differentiated with respect to w, giving

$$\theta_l(v(a_1, w))\, v_l'(w) = 0 . \tag{47}$$

It follows that the constant term vanishes in the expansion of the numerator of (45) in powers of $z - a_1$, so that this expansion begins with $z - a_1$ to the first power, at least. Since the expansion of the denominator begins with $(z - a_1)^2$, it follows that a_1 is, at most, a pole of the first order.

The same argument applies to $a_2, a_3, \cdots, a_{p-1}$. Therefore the rational function (43) is regular, except perhaps for first order poles at the $p - 1$ points $a_1, a_2, \cdots, a_{p-1}$.

Now, the theorem of Riemann-Roch: *a rational function on a Riemann surface of genus* p *has poles of a total order at least* $p + 1$, *unless the function reduces to a constant.*

(For elliptic functions, this is the well-known theorem that no non-constant elliptic function can have an order less than two.)

According to this theorem, the expression (43) must reduce to a constant, as to z; and therefore, by symmetry, a constant as to ζ; so that it is a function of w alone. *This establishes the preliminary identity* (41).

Using (42), the formula (41) can be rewritten, with regard to (43), in the form:

$$\begin{aligned}\Big\{ &\zeta_l(v(z,\zeta))\,\zeta_m(v(\zeta,w)) - \zeta_l(v(z,w))\,\zeta_m(v(z,\zeta)) \\ &\qquad - \zeta_l(v(z,w))\,\zeta_m(v(\zeta,w)) \\ &+ 2\pi i \frac{\partial \log\theta(v(z,\zeta))}{\partial \tau_{lm}} + 2\pi i \frac{\partial \log\theta(v(z,w))}{\partial \tau_{lm}} + 2\pi i \frac{\partial \log\theta(v(\zeta,w))}{\partial \tau_{lm}} \Big\} \\ &v_l'(w)\, v_m'(w) = \text{function } (w)\end{aligned} \tag{48}$$

(the factor 2 has been divided out, and the terms re-arranged).

If the reader will apply to this the operator

$$\frac{\partial^2}{\partial z\, \partial \zeta} \tag{49}$$

then—with perhaps the reminder that

$$\begin{aligned}\frac{\partial^2}{\partial z\,\partial\zeta}\frac{\partial \log\theta(v(z,\zeta))}{\partial\tau_{lm}} &= \frac{\partial}{\partial\tau_{lm}}\frac{\partial^2 \log\theta(v(z,\zeta))}{\partial z\,\partial\zeta} \\ &= \frac{\partial\,\varphi_{jk}(v(z,\zeta))}{\partial\tau_{lm}} v_j'(z)\, v_k'(\zeta) ,\end{aligned} \tag{50}$$

and with possible interchange of the summation indices l, m—he will easily obtain the fundamental identity (33).

8. **An example.** We conclude with an example of a contour that bounds a minimal surface of the topological type of a torus with one

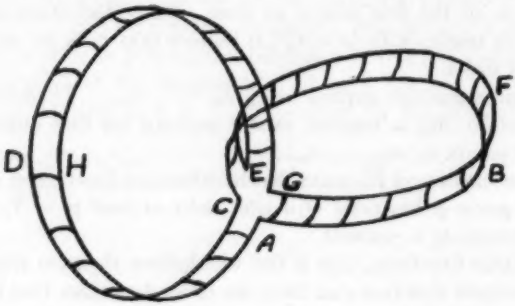

FIG. 1

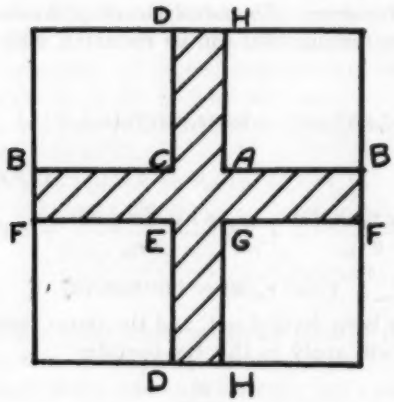

FIG. 2

perforation. This example was first given in our invited address[11] on the problem of Plateau to the American Mathematical Society in October, 1932.

[11] Published in *Bull. Amer. Math. Soc.*, v. 39 (1933), pp. 227–251, (but, to save space, this example and the accompanying figures were not included).

The figure represents a contour *ABCDEFGHA*, formed of four almost complete circular arcs, two in horizontal planes close together, the other two in vertical planes close together. If the soap-film experiment of Plateau be performed with this contour, a minimal surface will be obtained in the form, very nearly, of two catenoidal ribbons joined together. This minimal surface is topologically equivalent to the shaded region on the torus in figure 2, where we employ the usual model in the form of a square with opposite points on parallel sides regarded as identical. The unshaded region is simply-connected, and represents a (large) perforation in the torus.

If, for the horizontal circles, we use the soap-film consisting of two circular discs, and join to the catenoidal strip determined by the vertical circles, then we obtain a simply-connected minimal surface bounded by the same contour. Since the rôles of the horizontal and vertical circles can be interchanged, this simply-connected solution of the Plateau problem is evidently not unique. Obviously too, the film in the form of the perforated torus has smaller area than the simply-connected one.

The normal forms of two-dimensional figures of higher topological structure usually given in text-books on topology[13] afford further examples.

CAMBRIDGE, MASS.,
APRIL, 1936.

[13] See B. v.Kerékjártó, *Vorlesungen über Topologie*, (Berlin, 1923), p. 153, figures 31a, 31b.

AN APPLICATION OF KINETIC THEORY TO THE PROBLEMS OF EVAPORATION AND SUBLIMATION OF GASES HAVING MORE THAN ONE ATOM PER MOLECULE

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1. **Introduction.** From the standpoint of kinetic theory a polyatomic gas, by which we mean any gas having more than one atom per molecule, differs from a monatomic gas in that the molecular energy is made up not only of the translatory energy of the molecules, but also of the rotational energy, and the energy of internal atomic motions within the molecule. Thus according to classical theory if the total number of degrees of freedom of a molecule is f_c, this being the number of coordinates required to fix the molecule, it follows from the equipartition theorem that the mean kinetic energy per molecule is $\frac{1}{2}kTf_c$, where k is Boltzman's constant and T the absolute temperature. More generally if the number of squared terms in the expression for the energy of a molecule is f, the total mean energy per molecule, kinetic and potential, is $\frac{1}{2}kTf$.[1] It thus follows that the specific heat of the gas at constant volume is $\frac{1}{2}kf$, a constant. Although many gases are observed to have specific heats corresponding to $f = 3$ (monatomic gases), $f = 5$ (diatomic gases), and $f = 6$ over a considerable range of pressure and temperature; nevertheless in general the specific heats are functions of temperature and pressure, which fact calls attention to the insufficiency of the classical theory.

The defects of the classical theory are remedied by the quantum theory, according to which the possible energies associated with the rotations and the various internal motions are limited to discrete sets of values; and different molecules are in different states of excitation. It thus follows that there is no longer an equipartition of energy among the various coordinates; and the behavior of the specific heats no longer appears abnormal.

In a recent paper[2] the evaporation of monatomic gases was considered from the standpoint of kinetic theory; and the following results were obtained:

[1] Jeans, "Dynamical Theory of Gases", p. 80.

[2] Crout, "An Application of Kinetic Theory to the Problems of Evaporation and Sublimation of Monatomic Gases", *Journal of Mathematics and Physics*, vol. 15, p. 1.

a) An evaporating gas is not isotropic and homogeneous in the immediate vicinity of the liquid surface; hence in this vicinity the words "temperature" and "pressure" lose their ordinary significance. The state of the anisotropic gas at the liquid surface may, however, be expressed by means of two temperatures: a "longitudinal temperature" in the direction of evaporation, and a "transverse temperature" at right angles in this direction.

b) As the evaporating gas moves away from the liquid surface the molecules collide with each other, so that in the course of several mean free paths the gas is again isotropic; however, this final state of the gas differs from that of the stationary saturated vapor, the pressure, density, temperature, and drift velocity being functions of the liquid temperature and the rate of evaporation. These functions were derived and tabulated.

c) As the pressure into which the gas evaporates is lowered, the rate of evaporation and the drift velocity increase up to a certain critical point, at which the gas in its final isotropic state is moving with its sound velocity. If the discharge pressure is reduced below the critical value, the rate of evaporation and the constants of the evaporating gas retain their critical values.

The purpose of this paper is to extend the methods used for monatomic gases, and thus to obtain similar results for polyatomic gases. The theory is first developed retaining the classical concept of a gas (f = constant); then it is modified where necessary to bring it into accord with quantum conditions. This procedure is followed because it is simpler and more concise than would be the case if we were to consider all complications right from the start; also we are thus enabled to see where the classical theory leads.

We shall see that the curves for the isotropic gas data given by the classical theory approach definite limiting positions as f varies from five to infinity; thus in a sense we obtain a set of bands instead of a set of curves, which are applicable to all polyatomic gases. This picture is later found to be in accord with quantum conditions, and is probably the best that can be hoped for by way of a single set of data which is generally applicable. Once we have modified the classical theory to take care of the quantum conditions, and hence the peculiarities of individual gases, it follows naturally that a set of calculated data will apply only to a single gas. We shall see, however, that if we have available sets of data computed on the basis of classical theory for different values of f, this data is entirely sufficient to solve problems involving any gas, taking account of quantum conditions.

The calculations indicated by the formulas obtained by classical theory were carried out for diatomic gases ($f = 5$), and the results are placed in the form of tables and curves suitable for practical use. This data may be used without modification in the case of many diatomic gases.

The calculations required to obtain the limiting values of the constants of the isotropic gas as $f \rightarrow \infty$ were also carried out; and the results are also given in the form of tables and curves.

The analysis which follows applies directly and without modification to the process of sublimation; thus if throughout this paper we replace the word "evaporation" by "sublimation", and the word "liquid" by "solid", all methods and results apply in full to the sublimation of polyatomic gases.

Throughout this paper all units are C. G. S., and all temperatures are in degrees absolute, or Kelvin, except where it is expressly stated otherwise.

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2. Method of Attack. Fundamental Assumptions. We shall first build up the theory of evaporation *retaining the classical concept of a gas* (f = *constant*), which concept is characterised by the equipartition of energy. This will be done in Sections 2–10, after which the theory so developed will be modified to meet quantum conditions.

The fundamental assumption which is made concerning the mechanism of evaporation is that secondary emission is a neglible factor. By this we mean that the velocity distribution and the internal energy[3] distribution of the emitted molecules are not affected by those of the incident, or returning molecules. Thus in particular the velocity distribution and the internal energy distribution of the emitted molecules are the same as in the case where the liquid is in equilibrium with its saturated vapor, in which case the gas is isotropic and homogeneous. These distributions are thus the same as those of the molecules crossing

[3] We shall denote by "internal energy" all energy except the translatory energy of the molecules. The translatory energy will be referred to as "external energy".

from one side of any plane in the saturated vapor to the other side. The velocity distribution of the emitted molecules must thus be[4]

$$dN = \nu_0 \left(\frac{h_0 m}{\pi}\right)^{\frac{1}{2}} u\epsilon^{-h_0 m(u^2+v^2+w^2)}\, dudvdw, \qquad u \geqq 0, \tag{1}$$

as is shown in Section 3. Here u, v, and w are the velocity components parallel to the x, y, and z axes, respectively, of a set of Cartesian coordinates placed with the origin at the liquid surface and the positive x direction normal to the surface outward. dN is the number of molecules emitted per second per square centimeter whose centers of gravity have u, v, and w components of velocity lying between u and $u + du$, v and $v + dv$, and w and $w + dw$, respectively; furthermore

ν_0 = molecular density of saturated vapor, (2)
m = mass of molecule,
h_0 = temperature coefficient in Maxwellian velocity distribution of saturated vapor. ($h_0 = 3.65 \cdot 10.^{15} \div$ absolute temperature).

Similarly the internal energy[5] distribution of the emitted molecules must be

$$dE_i = \frac{f-3}{4h_0}\, dN, \tag{3}$$

where dE_i is the internal energy of the dN emitted molecules, and f is the total number of degrees of freedom of the molecule.[5] The number of internal degrees of freedom is thus $f - 3$. If (1) is substituted in (3), the internal energy distribution against u, v, and w is obtained. It may be noted that the coefficient of dN in (3) does not contain u, v, or w. It is evident that relations (1) and (3) completely specify the mechanical behavior of the liquid.

[4] It is assumed that the saturated vapor is sufficiently rare so that it may be treated as an ideal gas. This assumption is justified if we keep well away from that critical point above which the gas cannot be liquefied by pressure alone.

Eucken, "Lehrbuch der chemischen Physik", p. 230.

Schaefer, "Einführung in die theoretische Physik", vol. 2, p. 200.

The word "critical" as used in this paper has a meaning entirely different from that which it has in this footnote. Except where it is expressly stated otherwise, the word "critical" shall have that meaning assigned to it in Section 8.

[5] More exactly f is the number of squared terms in the expression for the total energy of a molecule; thus if a particular degree of freedom gives rise to two squared terms, then this degree of freedom must be counted twice in obtaining f. The relation (3) follows from the equipartition theorem.[1]

We now consider the one-dimensional[6] problem of a gas evaporating from a liquid, and adopt the following notation concerning gas flow across a unit plane area placed parallel to the liquid surface at a distance x from it:

M^+ = total mass of molecules crossing this area per second in the positive x direction;
M^- = total mass of molecules crossing this area per second in the negative x direction;
μ^+ = total x component of momentum of molecules crossing this area per second in the positive x direction;
μ^- = total x component of momentum of molecules crossing this area per second in the negative x direction; (4)
E_i^+, E_e^+, E^+ = total internal energy, total external energy, and total energy, respectively, of molecules crossing this area per second in the positive x direction;
E_i^-, E_e^-, E^- = total internal energy, total external energy, and total energy, respectively, of molecules crossing this area per second in the negative x direction.

Obviously these quantities are all functions of x, since the state of the gas depends upon x. For the liquid surface we also place

M_0^+ = total mass of molecules emitted per second per square centimeter;
μ_0^+ = total x component of momentum of molecules emitted per second per square centimeter; (5)
E_{0i}^+, E_{0e}^+, E_0^+ = total internal energy, total external energy, and total energy, respectively, of molecules emitted per second per square centimeter.

It is evident that the quantities (5) are determined by (1) and (3). Next we note that in the steady state the following necessary conditions are satisfied for any arbitrary closed surface fixed in space:

a) The total number of molecules entering the surface per second is zero.

[6] By "one-dimensional" we merely mean that the state of the gas varies only in the direction normal to the liquid surface. We do not infer that there are no components of molecular velocities parallel to this surface; in fact such parallel components are included in (1).

b) The total x component of momentum entering the surface per second is zero.
c) d) Same as b) except for y and z components of momentum, respectively. (6)
e) The total molecular energy entering the surface per second is zero.

Proceeding as in the case of a monotomic gas[2] we now apply the conditions (6) to the closed disc-shaped surface formed by the liquid surface, a parallel plane surface in the gas spaced a distance Δx from the liquid, and that portion of a right circular cylinder normal to these surfaces which is intercepted by them, the cross section of the cylinder being one square centimeter. It thus follows that[7]

$$\begin{aligned} M_0^+ &= (M^+)_{x=0} \\ \mu_0^+ &= (\mu^+)_{x=0} \\ E_0^+ &= (E^+)_{x=0} . \end{aligned} \tag{7}$$

The last equation of (7) may be written

$$E_{0i}^+ + E_{0e}^+ = (E_i^+)_{x=0} + (E_e^+)_{x=0} , \tag{8}$$

since

$$E^+ = E_i^+ + E_e^+ , \qquad E_0^+ = E_{0i}^+ + E_{0e}^+ . \tag{9}$$

We do not know the velocity distribution and internal energy distribution of the molecules of the gas just outside the liquid surface; hence proceeding as in the case of a monatomic gas we assume the following velocity distribution function

$$\nu \left(\frac{h_L h_T^2 m^3}{\pi^3} \right)^{\frac{1}{2}} e^{-m[h_L(u-u_0)^2 + h_T v^2 + h_T w^2]} . \tag{10}$$

This expression multiplied by $dudvdw$ gives the number of molecules per cubic centimeter of the anisotropic gas at the liquid surface whose centers of gravity have components of velocity lying between u and

[7] It is assumed that all molecules reaching the liquid surface from the gas enter the liquid; thus there is no reflection of such molecules at the liquid surface. According to our picture of evaporation, a molecule must have a certain energy in order to pass from the liquid to the gas in opposition to the surface attraction; however, a molecule need have no energy to pass from the gas to the liquid. In fact the surface attraction would tend to draw in every molecule reaching the surface from the outside, and thus prevent any reflection of such molecules.

$u + du$, v and $v + dv$, w and $w + dw$, respectively. The expression (10) differs from a Maxwellian distribution in that h_L differs from h_T. The distribution (10) contains the following four constants:

$$\begin{aligned} \nu &= \text{molecular density,} \\ h_L &= \text{longitudinal temperature coefficient,} \\ h_T &= \text{transverse temperature coefficient,} \\ u_0 &= \text{drift velocity,} \end{aligned} \tag{11}$$

which constants fix the state of the gas at the liquid surface, except for the internal energy of the molecules. It may be noted that if (10) is integrated with respect to u, v, and w, the limits in each case being $-\infty$ to ∞, the result is ν; also we see that (10) is even in v and w. As in the case of an ordinary isotropic gas we define

$$\begin{aligned} T_L &= \frac{1}{2h_L k} = \text{longitudinal temperature,} \\ T_T &= \frac{1}{2h_T k} = \text{transverse temperature,} \\ k &= \text{Boltzman's constant} = 1.371 \cdot 10.^{-16} \text{ ergs/degree.} \end{aligned} \tag{12}$$

For the internal energy we assume the distribution function

$$[f - 3]\left[\frac{1}{4h_0} + \frac{\gamma}{3}\left(\frac{1}{4h_L} + \frac{1}{2h_T} - \frac{3}{4h_0}\right)\right]. \tag{13}$$

This expression multiplied by dN gives the internal energy dE_i associated with the dN molecules of the anisotropic gas at the liquid surface. Noting that $(f - 3)$ is the number of internal degrees of freedom, and that $\left(\frac{1}{4h_L} + \frac{1}{2h_T}\right)$ is the mean external thermal energy per molecule of an anisotropic gas in the state (10), we see that (13) merely expresses the mean internal energy per molecule per degree of freedom as the sum of the mean thermal energy per molecule per degree of freedom of the stationary saturated vapor, $\frac{1}{4h_0}$, and an unknown number γ times the difference between the mean external thermal energy per molecule per external degree of freedom of the anisotropic gas, and the mean thermal energy per molecule per degree of freedom of the stationary saturated vapor. Since γ is left undetermined, the only assumption involved in (13) is that the internal energy distribution does not contain u, v, or w. The particular form given to expression (13) was chosen for convenience later on.

The constants ν, h_L, h_T, u_0 are determined as functions of the parameter γ by the three conditions (7), and the one condition that the rate of evaporation take a specified value. The state of the anisotropic gas at the liquid surface is thus fixed in terms of the parameter γ, which is a measure of the change in internal energy which the molecules undergo immediately after they are emitted from the liquid surface.

As the gas moves away from the surface, the molecules collide with each other; so that in the course of several mean free paths the gas in its final state is again isotropic, and thus has the Maxwellian velocity distribution

$$\nu_1 \left(\frac{h_1 m}{\pi}\right)^{\frac{3}{2}} e^{-h_1 m[(u-u_1)^2+v^2+w^2]}, \tag{14}$$

which is determined by the three constants

$$\begin{aligned} \nu_1 &= \text{molecular density,} \\ h_1 &= \text{temperature coefficient,} \\ u_1 &= \text{drift velocity.} \end{aligned} \tag{15}$$

Here again we have by definition

$$T_1 = \frac{1}{2h_1 k} = \text{temperature.} \tag{16}$$

Since the gas in its final isotropic state has an equipartition of energy, it follows that it has the internal energy distribution function

$$\frac{(f-3)}{4h_1}. \tag{17}$$

The constants (15) are determined by applying conditions a), b), and e) of (6) to the closed surface bounded by a cylinder of unit cross section normal to the liquid surface, and by two planes parallel to this surface: one in the gas just outside, and the other spaced far enough away so that it lies in gas which has reached its final isotropic state. Thus the final state of the gas for a given rate of evaporation is determined.

3. **Transfer of Mass, Momentum, and Energy Across Unit Area Normal to the Drift Velocity.** We now proceed to obtain the quantities (4) for the distribution functions (10) and (13). If we set up the integral expressions for the transfer of mass, momentum, and external energy across a unit area normal to the drift velocity, it is at once

evident that these expressions are identical with the corresponding expressions for a monatomic gas.[2] The resulting transfer equations for a polyatomic gas are therefore identical with the corresponding equations for a monatomic gas. Placing

$$\rho = u_0 \sqrt{h_L m}, \qquad \sigma = \frac{h_L}{h_T}, \qquad P(\rho) = \frac{2}{\sqrt{\pi}} \int_0^\rho \epsilon^{-x^2}\, dx = \text{Probability Integral}; \tag{18}$$

we thus have

$$\begin{aligned}
(M^+)_{z=0} &= \frac{m\nu}{2\sqrt{h_L m}} \left\{ \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho[1 + P(\rho)] \right\} \\
(M^-)_{z=0} &= \frac{m\nu}{2\sqrt{h_L m}} \left\{ \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho[1 - P(\rho)] \right\} \\
(\mu^+)_{z=0} &= \frac{m\nu}{2h_L m} \left\{ \frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} + \left[\rho^2 + \frac{1}{2}\right][1 + P(\rho)] \right\} \\
(\mu^-)_{z=0} &= \frac{m\nu}{2h_L m} \left\{ \frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} - \left[\rho^2 + \frac{1}{2}\right][1 - P(\rho)] \right\} \\
(E_e^+)_{z=0} &= \frac{m\nu}{4(h_L m)^{\frac{1}{2}}} \left\{ [\rho^2 + 1 + \sigma] \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)] \right\} \\
(E_e^-)_{z=0} &= \frac{m\nu}{4(h_L m)^{\frac{1}{2}}} \left\{ [\rho^2 + 1 + \sigma] \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 - P(\rho)] \right\}.
\end{aligned} \tag{19}$$

Noting (13) and the first two equations of (19), we also have

$$\begin{aligned}
(E_i^+)_{z=0} &= \frac{\nu}{2\sqrt{h_L m}} [f - 3]\left[\frac{1}{4h_0} + \frac{\gamma}{3}\left(\frac{1}{4h_L} + \frac{1}{2h_T} - \frac{3}{4h_0}\right)\right] \\
&\quad \times \left\{ \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho[1 + P(\rho)] \right\} \\
(E_i^-)_{z=0} &= \frac{\nu}{2\sqrt{h_L m}} [f - 3]\left[\frac{1}{4h_0} + \frac{\gamma}{3}\left(\frac{1}{4h_L} + \frac{1}{2h_T} - \frac{3}{4h_0}\right)\right] \\
&\quad \times \left\{ \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho[1 - P(\rho)] \right\}.
\end{aligned} \tag{20}$$

We still have the problem of determining the quantities (5), which problem is that of determining the transfer equations for the liquid; however, comparing (1) and (3) with (10) and (13), we see that the quantities (5) can be written down immediately by placing $\rho = 0$, $\sigma = 1$, $h_L = h_0$, $\nu = \nu_0$, and $\gamma = 0$ in (19) and (20). We thus have finally

$$\begin{aligned} M_0^+ &= \frac{m\nu_0}{2\sqrt{\pi h_0 m}} \\ \mu_0^+ &= \frac{m\nu_0}{4h_0 m} \\ E_{0\epsilon}^+ &= \frac{m\nu_0}{2\sqrt{\pi}(h_0 m)^{\frac{1}{2}}} \\ E_{0i}^+ &= \frac{\nu_0(f-3)}{8\sqrt{\pi m}\, h_0^{\frac{1}{2}}}. \end{aligned} \tag{21}$$

4. **Equilibrium Conditions—Liquid to Anisotropic Gas.** We shall now apply the equilibrium conditions (7), making use of the transfer equations (19), (20), and (21), just obtained; hence substituting, noting (8), we have

$$\begin{aligned} &\frac{m\nu_0}{2\sqrt{\pi h_0 m}} = \frac{m\nu}{2\sqrt{h_L m}}\left\{\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\,[1 + P(\rho)]\right\} \\ &\frac{m\nu_0}{4h_0 m} = \frac{m\nu}{2h_L m}\left\{\frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} + \left[\rho^2 + \frac{1}{2}\right][1 + P(\rho)]\right\} \\ &\frac{m\nu_0}{2\sqrt{\pi}\,(h_0 m)^{\frac{1}{2}}} + \frac{\nu_0(f-3)}{8\sqrt{\pi m}\, h_0^{\frac{1}{2}}} = \frac{m\nu}{4(h_L m)^{\frac{1}{2}}}\left\{[\rho^2 + 1 + \sigma]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}}\right. \\ &\quad \left. + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)]\right\} + \frac{\nu}{2\sqrt{h_L m}}[f-3] \\ &\quad \times \left[\frac{1}{4h_0} + \frac{\gamma}{3}\left(\frac{1}{4h_L} + \frac{1}{2h_T} - \frac{3}{4h_0}\right)\right]\left\{\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\,[1 + P(\rho)]\right\}. \end{aligned} \tag{22}$$

Noting the first equation of (22), we see that the last may be written

$$\frac{m\nu_0}{2\sqrt{\pi}(h_0 m)^{\frac{1}{2}}} + \frac{\gamma m\nu_0(f-3)}{8\sqrt{\pi}(h_0 m)^{\frac{1}{2}}} = \frac{m\nu}{4(h_L m)^{\frac{1}{2}}}\left\{[\rho^2 + 1 + \sigma]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}}\right.$$
$$\left. + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)]\right\} + \frac{\nu\gamma}{6\sqrt{h_L m}}[f-3] \qquad (23)$$
$$\times\left[\frac{1}{4h_L} + \frac{1}{2h_T}\right]\left\{\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\,[1 + P(\rho)]\right\}$$

We now place

$$\frac{\nu}{\nu_0} = \nu',$$

$$\frac{h_L}{h_0} = h_L', \quad \frac{h_T}{h_0} = h_T',$$

$$\phi_1(\rho) = \frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\,[1 + P(\rho)], \qquad (24)$$

$$\phi_2(\rho) = \frac{\rho\epsilon^{-\rho^2}}{\sqrt{\pi}} + \left[\rho^2 + \frac{1}{2}\right][1 + P(\rho)],$$

$$\phi_3(\rho, \sigma) = [\rho^2 + 1 + \sigma]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} + \rho\left[\rho^2 + \frac{3}{2} + \sigma\right][1 + P(\rho)];$$

hence substituting, (22) and (23) become

$$\frac{\nu'\sqrt{\pi}}{\sqrt{h_L'}}\phi_1 = 1$$

$$\frac{2\nu'}{h_L'}\phi_2 = 1 \qquad (25)$$

$$\frac{\nu'\sqrt{\pi}}{2h_L'^{\frac{1}{2}}}\left[\phi_3 + \frac{1}{3}\gamma(f-3)\left(\frac{1}{2} + \sigma\right)\phi_1\right] = 1 + \frac{1}{4}\gamma(f-3).$$

We now solve the equations (25) to obtain σ, h_L', and ν' in terms of ρ, γ, and f; hence dividing the last two equations of (25) by the first, we eliminate ν'; thus

$$\frac{2}{\sqrt{\pi h_L'}}\left(\frac{\phi_2}{\phi_1}\right) = 1 \qquad (26)$$

$$\frac{1}{2h_L'\phi_1}\left[\phi_3 + \frac{1}{3}\gamma(f-3)\left(\frac{1}{2} + \sigma\right)\phi_1\right] = 1 + \frac{1}{4}\gamma(f-3).$$

Dividing the second equation of (26) by the first squared, we elimminate h_L'; hence

$$\frac{\pi\phi_1}{8\phi_2^2}\left[\phi_3 + \frac{1}{3}\gamma(f-3)\left(\frac{1}{2}+\sigma\right)\phi_1\right] = 1 + \frac{1}{4}\gamma(f-3),$$

or

$$\phi_3 + \frac{1}{3}\gamma(f-3)\left(\frac{1}{2}+\sigma\right)\phi_1 = \frac{8\phi_2^2}{\pi\phi_1}\left[1 + \frac{1}{4}\gamma(f-3)\right].$$

Noting (24) we see that this expression may be written

$$\phi_1\sigma + \frac{1}{3}\gamma(f-3)\,\phi_1\sigma = \frac{2\phi_2^2}{\pi\phi_1}[4 + \gamma(f-3)]$$
$$- \frac{1}{6}\gamma(f-3)\,\phi_1 - [\rho^2+1]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho\left[\rho^2+\frac{3}{2}\right][1+P(\rho)];$$

whence

$$\sigma = \left\{\frac{2\phi_2^2}{\pi\phi_1}[4+\gamma(f-3)] - \frac{1}{6}\gamma(f-3)\,\phi_1 - [\rho^2+1]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}}\right.$$
$$\left. - \rho\left[\rho^2+\frac{3}{2}\right][1+P(\rho)]\right\} \div \left[1+\frac{1}{3}\gamma(f-3)\right]\phi_1, \tag{27}$$

an expression which gives σ explicitly in terms of ρ and the parameter γ for a given value of f.

From the first equation of (26) we have at once

$$h_L' = \frac{4}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2. \tag{28}$$

From (12) and (18) we also have

$$h_T' = \frac{h_L'}{\sigma}$$
$$\frac{T_L}{T_0} = \frac{1}{h_L'} \tag{29}$$
$$\frac{T_T}{T_0} = \frac{\sigma}{h_L'}.$$

From (28) and the first equation of (25) we now have

$$\nu' = \frac{2}{\pi}\left(\frac{\phi_2}{\phi_1^2}\right). \tag{30}$$

The rate of evaporation per unit area of liquid is obviously

$$M = (M^+)_{x=0} - (M^-)_{x=0} = m\nu u_0 = \frac{m\nu\rho}{\sqrt{h_L m}};$$

whence it follows from (21), (28), and (30) that

$$\frac{M}{M_0^+} = \frac{2\rho}{\phi_1}. \tag{31}$$

We shall finally obtain the pressure p_s acting on the liquid surface; thus from the second equation of (7) we have at once

$$p_s = \mu_0^+ - (\mu^-)_{x=0} = (\mu^+)_{x=0} - (\mu^-)_{x=0};$$

whence substituting from (19) it follows that

$$p_s = \frac{\nu}{h_L}\left(\rho^2 + \frac{1}{2}\right). \tag{32}$$

The saturation vapor pressure may be obtained from (32) by placing $\nu = \nu_0$, $h_L = h_0$, and $\rho = 0$; thus

$$p_0 = \frac{\nu_0}{2h_0}. \tag{33}$$

Dividing (32) by (33), noting the second equation of (25), we have finally

$$\frac{p_s}{p_0} = \frac{1}{\phi_2}\left(\rho^2 + \frac{1}{2}\right). \tag{34}$$

If desired, the drift velocity may be obtained from (18). We see that equations (27), (28), (29), (30), (31), and (34) give the various constants of the anisotropic gas in terms of those of the saturated vapor and the parameters ρ, γ, and f. It may be noted that *the expressions for* $\frac{h_L}{h_0}, \frac{T_L}{T_0}, \frac{\nu}{\nu_0}, \frac{M}{M_0^+}$, *and* $\frac{p_s}{p_0}$ *are independent of* f *and* γ; *and are identical with the corresponding expressions for a monatomic gas.*

5. **Calculations, Tables, and Curves for the Anisotropic Gas.** In Section 4 we obtained expressions for the constants of the anisotropic gas in terms of those of the stationary saturated vapor, the two parameters ρ and γ, and the number of degrees of freedom[5] of the molecule, f. In the case of diatomic gases, $f = 5$, the calculations indicated by the above expressions have been carried out; and will now be described.

The first step was to choose suitable sets of values for ρ and γ; however, in view of previous experience with monatomic gases it was not necessary to carry out a set of preliminary calculations. Instead, the set of values of ρ used for monatomic gases was used here, since ϕ_1, ϕ_2, $\frac{h_L}{h_0}$, $\frac{T_L}{T_0}$, $\frac{\nu}{\nu_0}$, $\frac{M}{M_0^+}$, and $\frac{p_s}{p_0}$ are the same for both monatomic and polyatomic gases regardless of f and γ; and also, as we shall see, because we are primarily interested in values of ρ which are smaller than .46.

The parameter γ was arbitrarily given the values 0, .5, and 1. The value $\gamma = 0$ corresponds to the case where the mean internal energy per molecule of the anisotropic gas is the same as the mean internal energy per molecule emitted from the liquid; thus by placing $\gamma = 0$, we find what happens if the mean internal energy is not altered as the molecules pass from the liquid to the anisotropic gas. Similarly the value $\gamma = 1$ corresponds to a mean internal energy[8] per degree of freedom[8] per molecule of the anisotropic gas equal to the mean external thermal energy per degree of freedom per molecule, as we shall see in Section 6. We thus see that in a sense the range $0 \leq \gamma \leq 1$ covers a set of possibilities ranging between no adjustment and complete adjustment of the internal molecular energy. It thus follows that the value $\gamma = .5$ corresponds to a state of partial adjustment of the internal molecular energy.

Values of $\frac{2}{\sqrt{\pi}} \int_0^\rho \epsilon^{-x^2}\, dx$ and $\frac{2}{\sqrt{\pi}} \epsilon^{-\rho^2}$ were taken from a paper by Burgess.[8] The calculations were carried out on a Marchant computing machine capable of handling nine place factors. All machine calculations were carried through three times, for each of which the recorded value was checked two times. The calculations themselves merely consisted in carrying out the operations indicated in Section 4. The results are given in Tables I and II, and in Figs. 1, 2, 3, 4, 5, and 6. The curves Figs. 5 and 6 were obtained by plotting $\frac{p_s}{p_0}$, $\frac{T_L}{T_0}$, and $\frac{T_T}{T_0}$,

[8] Burgess, "On the Definite Integral $\frac{2}{\sqrt{\pi}} \int_0^t \epsilon^{-t^2}\, dt$, with Extended Tables of Values", *Transactions of the Royal Society of Edinburgh*, vol. 39, p. 257.

[9] By this statement we mean that the difference between a value tabulated and the true value amounts to less than one unit in the last decimal place. That such a statement can be made follows from the fact that a set of calculations involving differences was made paralleling the machine calculations here described, by means of which bounds upon the error were obtained at each step.

respectively, against $\frac{M}{M_0^+}$ for the same values of ρ, thus eliminating ρ. All figures tabulated are significant.[9]

6. **Partition of Energy in the Anisotropic Gas. Significance of Two Temperatures.** The mean energy per molecule of the anisotropic gas in the state (10) and (13) is given by the expression

$$\left(\frac{h_L h_T^2 m^3}{\pi^3}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{[f-3]\left[\frac{1}{4h_0}+\frac{\gamma}{3}\left(\frac{1}{4h_L}+\frac{1}{2h_T}-\frac{3}{4h_0}\right)\right]\right.$$
$$\left.+\frac{m}{2}(u^2+v^2+w^2)\right\} e^{-m[h_L(u-u_0)^2+h_T v^2+h_T w^2]}\,dudvdw. \tag{35}$$

This integral can be split into two integrals corresponding to the two terms in the brace. The first of these may be evaluated quite simply since the first term in the brace does not contain u, v, or w; the second integral is identical with the corresponding expression for a monatomic gas. It thus follows that the mean energy per molecule of the anisotropic gas is

$$\frac{1}{2}mu_0^2+\frac{1}{4h_L}+\frac{1}{2h_T}+[f-3]\left[\frac{1}{4h_0}+\frac{\gamma}{3}\left(\frac{1}{4h_L}+\frac{1}{2h_T}-\frac{3}{4h_0}\right)\right],$$

or

$$\frac{1}{2}mu_0^2+\frac{1}{2}kT_L+kT_T$$
$$+[f-3]\left[\frac{1}{2}kT_0+\frac{\gamma}{3}\left(\frac{1}{2}kT_L+kT_T-\frac{3}{2}kT_0\right)\right]. \tag{36}$$

From (36) we see that the mean energy per molecule may be considered as made up of the drift energy $\frac{1}{2}mu_0^2$, the thermal energy $\frac{1}{2}kT_L$ for the x coordinate, or degree of freedom, the thermal energies $\frac{1}{2}kT_T$ each for the y and z coordinates, and the energy $\left[\frac{1}{2}kT_0+\frac{\gamma}{3}\left(\frac{1}{2}kT_L+kT_T-\frac{3}{2}kT_0\right)\right]$ for each of the $(f-3)$ internal[3] degrees of freedom,[5] respectively. We thus see that the transverse and longitudinal temperatures are a direct measure of the mean molecular energy for the corresponding coordinates, or degrees of freedom, respectively; hence these temperatures have the same significance as has the single temperature of an isotropic gas.

Referring to Fig. 6, we see that for a given rate of evaporation anywhere in the range $0 \leq \frac{M}{M_0^+} < .81$, which range, we shall see later, is that in which we are primarily interested, the percentage deviation of T_T from T_0 is not only small, but is also much smaller than the corresponding deviation of T_L. Noting (36) we thus infer that the y and z degrees of freedom tend to remain isolated from the x degree of freedom, along which the evaporation occurs, during the process of evaporation. If, now, we assume that the $(f - 3)$ internal degrees of freedom likewise tend to remain isolated, then it follows that γ should be approximately zero. In view of this and the comments in Section 5, we should expect γ to lie between zero and one; and to approximate zero. We shall see, however, that it is not necessary to know γ in order to determine the final isotropic state of the gas.

7. **Equilibrium Conditions—Anisotropic to Isotropic Gas.** In Section 4 we determined the state of the anisotropic gas at the liquid surface. Our next step is to determine the state of the gas at a distance of several mean free paths from the surface, where it has become isotropic. We shall thus determine the constants (15) of the distributions (14) and (17) in the manner described in Section 2. Let us denote by x_1 the abscissa of the right hand boundary of the closed surface, which boundary is the plane lying in the isotropic gas; then noting (4) we see that conditions a, b, and e of (6) may be written

$$\begin{aligned}
(M^+)_{x=x_1} - (M^-)_{x=x_1} &= (M^+)_{x=0} - (M^-)_{x=0} \\
(\mu^+)_{x=x_1} - (\mu^-)_{x=x_1} &= (\mu^+)_{x=0} - (\mu^-)_{x=0} \\
(E_e^+)_{x=x_1} - (E_e^-)_{x=x_1} + (E_i^+)_{x=x_1} - (E_i^-)_{x=x_1} & \\
= (E_e^+)_{x=0} - (E_e^-)_{x=0} + (E_i^+)_{x=0} - (E_i^-)_{x=0}&.
\end{aligned} \tag{37}$$

The right hand sides of these equations may be obtained from (19) and (20), the result being the right hand sides of (39), respectively. Comparing (14) and (17) with (10) and (13), we see that the left hand sides of (37) may be obtained by placing $x = x_1$, $\nu = \nu_1$, $h_L = h_1$, $h_T = h_1$, $h_0 = h_1$, $\sigma = 1$, and by writing

$$\rho_1 = u_1 \sqrt{h_1 m} \tag{38}$$

in place of ρ in the right hand sides of (39), respectively, the result being

the left hand sides of (39). The equilibrium conditions (37) thus become

$$\begin{gathered}
\frac{m\nu_1\,\rho_1}{\sqrt{h_1 m}} = \frac{m\nu\,\rho}{\sqrt{h_L m}} \\
\frac{m\nu_1}{h_1 m}\left(\rho_1^2 + \frac{1}{2}\right) = \frac{m\nu}{h_L m}\left(\rho^2 + \frac{1}{2}\right) \\
\frac{m\nu_1\rho_1}{2(h_1 m)^{\frac{1}{2}}}\left(\rho_1^2 + \frac{5}{2}\right) + \frac{m\nu_1\,\rho_1}{4(h_1 m)^{\frac{1}{2}}}(f-3) = \frac{m\nu\,\rho}{2(h_L m)^{\frac{1}{2}}}\left(\rho^2 + \frac{3}{2} + \sigma\right) \\
+ \frac{\nu\rho}{\sqrt{h_L m}}[f-3]\left[\frac{1}{4h_0} + \frac{\gamma}{3}\left(\frac{1}{4h_L} + \frac{1}{2h_T} - \frac{3}{4h_0}\right)\right].
\end{gathered} \tag{39}$$

Placing

$$\begin{aligned}
\nu_1' &= \frac{\nu_1}{\nu}, \\
h_1' &= \frac{h_1}{h_L},
\end{aligned} \tag{40}$$

the equations (39) become

$$\begin{gathered}
\frac{\nu_1'\rho_1}{\sqrt{h_1'}} = \rho \\
\frac{\nu_1'}{h_1'}\left(\rho_1^2 + \frac{1}{2}\right) = \left(\rho^2 + \frac{1}{2}\right) \\
\frac{\nu_1'\,\rho_1}{h_1'^{\frac{1}{2}}}\left(\rho_1^2 + 1 + \frac{1}{2}f\right) = \rho\left\{\rho^2 + \sigma\left[1 + \frac{\gamma}{3}(f-3)\right]\right. \\
\left. + \frac{h_L'}{2}(1-\gamma)(f-3) + \frac{\gamma}{6}(f-3) + \frac{3}{2}\right\}.
\end{gathered} \tag{41}$$

We shall now solve the equations (41) to obtain ν_1', h_1', and ρ_1 in terms of ρ, γ, and f; hence dividing the last two equations by the first, we have

$$\begin{gathered}
\frac{1}{\sqrt{h_1'}}\left(\frac{\rho_1^2 + \frac{1}{2}}{\rho_1}\right) = \left(\frac{\rho^2 + \frac{1}{2}}{\rho}\right) \\
\frac{1}{h_1'}\left(\rho_1^2 + 1 + \frac{1}{2}f\right) = \rho^2 + \sigma\left[1 + \frac{\gamma}{3}(f-3)\right] \\
+ \frac{h_L'}{2}(1-\gamma)(f-3) + \frac{\gamma}{6}(f-3) + \frac{3}{2}.
\end{gathered} \tag{42}$$

Dividing the second equation of (42) by the first squared, we have

$$\frac{\rho_1^2(\rho_1^2 + 1 + \frac{1}{2}f)}{(\rho_1^2 + \frac{1}{2})^2} = \alpha\,, \tag{43}$$

where

$$\alpha = \frac{\rho^2}{(\rho^2 + \frac{1}{2})^2}\left\{\rho^2 + \sigma\left[1 + \frac{\gamma}{3}(f-3)\right] + \frac{h_L'}{2}(1-\gamma)(f-3) + \frac{\gamma}{6}(f-3) + \frac{3}{2}\right\}. \tag{44}$$

Substituting (27) and (28) in (44), it follows that

$$\alpha = \frac{\rho^2}{(\rho^2 + \frac{1}{2})^2}\left[\rho^2 + \frac{1}{\phi_1}\left\{\frac{2\phi_2^2}{\pi\phi_1}[4 + \gamma(f-3)] - \frac{1}{6}\gamma(f-3)\phi_1 - [\rho^2+1]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho\left[\rho^2 + \frac{3}{2}\right][1 + P(\rho)]\right\}\right. \tag{45}$$
$$\left. + \frac{2}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2(1-\gamma)(f-3) + \frac{1}{6}\gamma(f-3) + \frac{3}{2}\right].$$

The coefficient of γ in the large bracket of (45) is

$$[f-3]\left[\frac{2}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2 - \frac{1}{6} - \frac{2}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2 + \frac{1}{6}\right],$$

which is zero. It thus follows that

$$\alpha = \frac{\rho^2}{(\rho^2 + \frac{1}{2})^2}\left[\rho^2 + \frac{3}{2} + \sigma_m + \tfrac{1}{2}h_L'(f-3)\right], \tag{46}$$

where

$$\sigma_m = \frac{1}{\phi_1}\left\{\frac{8\phi_2^2}{\pi\phi_1} - [\rho^2+1]\frac{\epsilon^{-\rho^2}}{\sqrt{\pi}} - \rho\left[\rho^2 + \frac{3}{2}\right][1 + P(\rho)]\right\}. \tag{47}$$

Comparing (47) with (27), we see that σ_m is obtained either by placing $\gamma = 0$ or by placing $f = 3$ in (27). It thus follows that σ_m is the value of σ which would be obtained for a monatomic gas. *We see from* (46) *that α is independent of γ*; also we see that f appears only in the last term. Attention is called to the fact that σ_m and h_L' are functions of ρ which were met in dealing with monatomic gases, and which have therefore been computed and tabulated.

Returning to (43), we see that that expression may be written

$$(1-\alpha)\,\rho_1^4 + (1+\tfrac{1}{2}f-\alpha)\,\rho_1^2 - \tfrac{1}{4}\alpha = 0\,. \tag{48}$$

Solving this biquadratic for ρ_1^2, we have

$$\rho_1^2 = \frac{\alpha - 1 - \tfrac{1}{2}f \pm \sqrt{(1+\tfrac{1}{2}f)^2 - \alpha(1+f)}}{2(1-\alpha)}\,, \tag{49}$$

an expression which gives ρ_1 explicitly in terms of ρ for a given value of f.

The right hand side of the second equation of (42) is identical with the expression in the brace of (44), which was shown to be independent of γ, and equal to

$$\rho^2 + \tfrac{3}{2} + \sigma_\infty + \tfrac{1}{2}\,h_L'\,(f-3)\,; \tag{50}$$

hence transposing, we have

$$h_1' = \frac{\rho_1^2 + 1 + \tfrac{1}{2}f}{\rho^2 + \tfrac{3}{2} + \sigma_\infty + \tfrac{1}{2}\,h_L'\,(f-3)}\,. \tag{51}$$

We may also express h_1' in the form

$$h_1' = \left[\frac{\rho\,(\rho_1^2 + \tfrac{1}{2})}{(\rho^2 + \tfrac{1}{2})\,\rho_1}\right]^2, \tag{52}$$

using the first equation of (42). This expression together with (28) gives $\dfrac{h_1}{h_0}$ in terms of ρ and f, for

$$\frac{h_1}{h_0} = h_1' h_L'\,. \tag{53}$$

The temperature ratio

$$\frac{T_1}{T_0} = \frac{h_0}{h_1} \tag{54}$$

now follows immediately from (53).

The pressure ratio is obtained by dividing the relations

$$p_1 = \nu_1 k T_1$$

$$p_0 = \nu_0 k T_0\,;$$

thus noting (54) we have

$$\frac{p_1}{p_0} = \frac{\nu_1 h_0}{\nu_0 h_1} = \frac{\nu' \nu_1'}{h_L' h_1'}\,; \tag{55}$$

whence

$$\frac{p_1}{p_0} = \frac{1}{2\phi_2}\left(\frac{\rho^2 + \frac{1}{2}}{\rho_1^2 + \frac{1}{2}}\right), \tag{56}$$

due to (28), (30), and the second equation of (41). Noting (34), we see that (56) may be written

$$\frac{p_1}{p_0} = \left[\frac{1}{2(\rho_1^2 + \frac{1}{2})}\right]\frac{p_s}{p_0}. \tag{57}$$

The relation (55) may be written

$$\frac{\nu_1}{\nu_0} = \left(\frac{h_1}{h_0}\right)\left(\frac{p_1}{p_0}\right), \tag{58}$$

an expression which gives $\frac{\nu_1}{\nu_0}$ in terms of ρ and f by means of (53) and (57).

The drift velocity u_1 may, if desired, be obtained from (38), (49), and (53). We have now obtained the various constants of the isotropic gas in terms of ρ for a given value of f. It may be noted that these constants are independent of γ.

8. **Physical Significance of ρ and ρ_1. Limiting Value of Drift Velocity.** The quantities ρ and ρ_1 were introduced by (18) and (38) merely to simplify the transfer equations (19) and (39), and were given no physical significance whatsoever; however, we may write

$$\rho^2 = \tfrac{1}{2}\left(\frac{\frac{mu_0^2}{2}}{\frac{1}{4h_L}}\right)$$
$$\rho_1^2 = \tfrac{1}{2}f\left(\frac{\frac{mu_1^2}{2}}{\frac{f}{4h_1}}\right); \tag{59}$$

whence, noting (36), we see that ρ^2 is one half the ratio of the drift energy per molecule to the mean longitudinal thermal energy per molecule of the anisotropic gas in the state (10) and (13). Also we see that ρ_1^2 is $\frac{1}{2}f$ times the ratio of the drift energy per molecule to the mean thermal energy per molecule of the isotropic gas in the state (14) and (17).

An additional significance may be given to ρ_1 by noting that the velocity of sound in the isotropic gas is

$$u_s = \sqrt{\frac{f+2}{2fh_1m}}, \tag{60}$$

since the ratio of the specific heats of a polyatomic gas is $\frac{f+2}{f}$. It thus follows from (38) that

$$\rho_1 = \frac{u_1}{u_s}\sqrt{\frac{f+2}{2f}}. \tag{61}$$

We thus see that ρ_1 *is* $\sqrt{\frac{f+2}{2f}}$ *times the ratio of the drift velocity to the velocity of sound in the isotropic gas.*

We shall next consider more closely the relationship between α and ρ_1, which is given either by (43) or by (49). In view of (61) and the physical meaning of ρ_1, it is apparent that we are interested only in real positive values of ρ_1; furthermore we see from (43) that α is a continuous one-valued function of ρ_1 with continuous derivatives in the region $\rho_1 \geqq 0$ for any fixed value of f. Due to the physical significance of f, however, it is evident that we are interested only in the values $f \geqq 3$, the equality sign corresponding to monatomic gases. Noting (43) we see that

$$\begin{gathered} \alpha = 0 \quad \text{when} \quad \rho_1 = 0, \\ \lim_{\rho_1 \to \infty} \alpha = 1, \end{gathered} \tag{62}$$

for any fixed value of f.

Placing $d\alpha/d\rho_1$ equal to zero, we obtain the expression

$$\rho_1[1 + \tfrac{1}{2}f - \rho_1^2 f] = 0,$$

after cancelling out factors which do not affect the real positive roots. It thus follows that the slope of the curve of α against ρ_1 is zero at the origin; also it follows that there is but one positive value of ρ_1 for which the slope vanishes, this being

$$\rho_1 = \sqrt{\frac{f+2}{2f}}. \tag{63}$$

Placing this value of ρ_1 in the expression for $d^2\alpha/d\rho_1^2$ obtained by differ-

entiating (43) twice, we find that the sign of the resulting expression is that of the quantity

$$-\frac{1}{f}(f^2 + 3f + 2).$$

Since f is positive it thus follows that $d^2\alpha/d\rho_1^2$ is negative for the one positive value of ρ_1 for which $d\alpha/d\rho_1 = 0$; hence we have a maximum at this point. Similarly if we place $\rho_1 = 0$ in the expression for $d^2\alpha/d\rho_1^2$ it follows that the sign of the resulting expression is that of the quantity $(f + 2)$, which is positive; hence we have a minimum at the origin.

The last two paragraphs lead to the conclusion that for any value of $f \geqq 3$ the curve of α against ρ_1 starts with a minimum at the origin, rises and takes its maximum value when ρ_1 reaches the value given by (63), then falls and approaches the asymptote $\alpha = 1$. It is to be noted that we are interested only in the values $\rho_1 \geqq 0$, and in the values $f \geqq 3$. For any such value of f we thus have a curve of the form Fig. 7.[10]

The maximum value of α is given by the expression

$$\alpha_c = \frac{(1 + \frac{1}{2}f)^2}{1 + f}, \tag{64}$$

which is obtained either by substituting (63) in (43), or by placing the expression under the radical in (49) equal to zero. It thus follows that ρ is limited to those values for which

$$0 \leqq \alpha \leqq \alpha_c. \tag{65}$$

We shall refer to the highest point on the curve of α against ρ_1 as the *critical point*, and to the data corresponding to this point as the *critical data*. The coordinates of the critical point are thus given by (63) and (64). We shall now investigate the stability of the two branches into which the curve is divided by the critical point.

It may readily be verified that the expression $\left[\frac{\rho^2}{(\rho^2 + \frac{1}{2})^2}\right]$ and its derivative are both positive in the interval $0 < \rho < .7$; also, referring to Table I we see that the same is true of the expression (50) for $f \geqq 3$. Noting (46) it thus follows that

$$\frac{d\alpha}{d\rho} > 0, \qquad 0 < \rho < .7, f \geqq 3; \tag{66}$$

[10] Referring to (49) we see that for the plus sign both numerator and denominator vanish for $\alpha = 1$ regardless of f; however, applying L'Hospital's rule for indeterminate forms, we find that $\rho_1 \to \frac{1}{\sqrt{2f}}$ as $\alpha \to 1$.

whence it follows that

$$\frac{d}{d\alpha}\left(\frac{M}{M_0^+}\right) > 0, \qquad 0 < \rho < .7, f \geqq 3; \tag{67}$$

for it is evident from Table I that $d(M/M_0^+)/d\rho$ is positive in this interval.

We shall next show that the critical value of ρ is smaller than .55 if $f \geqq 3$. We have

$$\frac{(1 + \frac{1}{2}f)^2}{1 + f} < \frac{.30}{.66}[3.6 + \tfrac{1}{2}\,1.7(f - 3)] \tag{68}$$

if

$$(1 + \tfrac{1}{2}f)^2 < (.45 + .38f)(1 + f),$$

which in turn is true if

$$1 + f + \tfrac{1}{4}f^2 < .45 + .83f + .38f^2,$$

or

$$0 < -.55 - .17f + .13f^2.$$

But this relation is true for $f \geqq 3$, since both roots of the quadratic are smaller than three; hence (68) is established for $f \geqq 3$. The left hand side of (68) is α_c; also it may be verified by Table I that the right side of (68) is smaller than the value obtained by placing $\rho = .55$ in (46). It thus follows that

$$\alpha_c < (\alpha)_{\rho=.55}, \qquad f \geqq 3; \tag{69}$$

hence, noting (65) and (66), and noting from (46) that $\alpha = 0$ when $\rho = 0$, we have

$$0 \leqq \rho \leqq \rho_c < .55, \qquad f \geqq 3, \tag{70}$$

ρ_c being the critical value of ρ. The relations (66) and (67) are thus established for all parts of the curve Fig. 7.

The condition that operation at any point of the curve be stable is that

$$\frac{d\left(\frac{p_1}{p_0}\right)}{d\left(\frac{M}{M_0^+}\right)} < 0, \tag{71}$$

which merely states that a slight *decrease* in pressure must give rise to a corresponding *increase* in the rate of evaporation. Noting (57) we see that the left hand side of (71) may be written

$$\frac{\left(\rho_1^2+\frac{1}{2}\right)\frac{d}{d\rho}\left(\frac{p_s}{p_0}\right)-2\rho_1\frac{p_s}{p_0}\frac{d\rho_1}{d\alpha}\frac{d\alpha}{d\rho}}{\left[\frac{d}{d\rho}\left(\frac{M}{M_0^+}\right)\right]\left[2\left(\rho_1^2+\frac{1}{2}\right)^2\right]}, \tag{72}$$

the sign of which is that of the expression

$$\left(\rho_1^2+\frac{1}{2}\right)\frac{d}{d\rho}\left(\frac{p_s}{p_0}\right)-2\rho_1\frac{p_s}{p_0}\frac{d\rho_1}{d\alpha}\frac{d\alpha}{d\rho}, \tag{73}$$

since $\frac{d}{d\rho}\left(\frac{M}{M_0^+}\right) > 0$, as is seen from Table I. From this table and (66) we also see that $\frac{d}{d\rho}\left(\frac{p_s}{p_0}\right) < 0$ and $\frac{d\alpha}{d\rho} > 0$ for $0 < \rho \leqq \rho_0$.

For the portion of the curve Fig. 7 which lies between the origin and the critical point, we have $\frac{d\rho_1}{d\alpha} > 0$; hence both terms of (73) are negative, (71) is satisfied; and this branch constitutes a stable operating region.

At a point on the curve to the right of the critical point $d\rho_1/d\alpha < 0$, and the two terms of (73) are of opposite sign; hence the resulting sign of the whole expression is not apparent. If, however, we move along the curve and approach the critical point, the factors of the left hand term of (73) approach definite finite limits, as do those of the right hand term with the exception of $d\rho_1/d\alpha$. Since $d\rho_1/d\alpha$ approaches negative infinity, it follows that the right hand term will predominate, and (73) will be positive when we have approached within a certain distance of the critical point. Noting (71) we thus see that there is an unstable region lying immediately to the right of the critical point, and bounded on the left by this point.

As we approach the critical point from either the right or the left, the quantities $\left(\rho_1^2+\frac{1}{2}\right)$, $\frac{d}{d\rho}\left(\frac{M}{M_0^+}\right)$, and the factors of the two terms of (73) referred to in the preceding paragraph all approach definite finite limits; however, $\frac{d\rho_1}{d\alpha} \to -\infty$ if the approach is from the right, whereas $\frac{d\rho_1}{d\alpha} \to +\infty$ if the approach is from the left. It thus follows from (72)

and (73) that if we move along the curve Fig. 7 toward the critical point, then

$$\frac{d\left(\frac{p_1}{p_0}\right)}{d\left(\frac{M}{M_0^+}\right)} \to -\infty \text{ if the approach is from the left,}$$
$$\frac{d\left(\frac{p_1}{p_0}\right)}{d\left(\frac{M}{M_0^+}\right)} \to +\infty \text{ if the approach is from the right.} \tag{74}$$

These relations are true for any value of $f \geqq 3$.

If we place $\rho = 0$ in the various formulas of Sections 4 and 7, it may be readily verified that regardless of f we obtain the numerical data given in the first row of Tables I, II, and III, which obviously corresponds to the stationary saturated vapor, and to the origin of the curve Fig. 7. It is thus evident that as we increase the rate of evaporation from zero by decreasing the discharge pressure below p_0, the operating point on the curve moves from the origin up along the left branch, which we have seen is stable, and finally reaches the critical point, this point being attainable from the left by virtue of (71) and the first equation of (74). It follows from the second equation of (74) and the fact that there is an unstable region immediately to the right of the critical point, which fact is also expressed by (74), that *for any value of* $f \geqq 3$ *it is physically impossible to pass beyond the critical point; hence the entire right hand branch of the curve Fig. 7 is physically unattainable.* We are thus interested only in the plus sign in equation (49).[10]

When operation is at the critical point, ρ_1 is given by (63). This value substituted in (61) gives

$$u_1 = u_s, \tag{75}$$

from which we see that the isotropic gas is moving with its sound velocity.

The last few paragraphs lead us to the following picture: *if an evaporating gas moves through a distance sufficient to allow the gas to become isotropic;*[11] *then* ρ *is restricted to the range* (70), *the lower bound*

[11] This condition requires that the gas encounter no obstacles or condensing surfaces within a distance of several mean free paths of the liquid surface. See Section 9.

corresponding to the stationary saturated vapor; the upper bound being the critical point, at which the rate of evaporation takes its maximum value, and at which the isotropic gas moves with its sound velocity.[12] Since the right hand branch of the curve Fig. 7 is unattainable, it follows from (75) and (61) that *the drift velocity of the isotropic gas can never exceed its sound velocity.*

These results involving the velocity of sound may be shown to be independent of the form of the velocity distribution and the internal energy distribution assumed for the gas at the liquid surface (in this paper the functions (10) and (13)), and may be shown to depend only upon (14) and (17), which are known to be exact, since the gas is isotropic at the point $x = x_1$. Thus in place of (10) let us assume the arbitrary velocity distribution

$$f(u, v, w, \beta_1, \beta_2, \cdots, \beta_n), \tag{76}$$

where $\beta_1, \beta_2, \cdots, \beta_n$ are constants which fix the state of the gas; and in place of (13) let us assume the arbitrary internal energy distribution

$$g(u, v, w, \beta_1, \beta_2, \cdots, \beta_n). \tag{77}$$

Carrying through the discussion as before, we have in place of (39)

$$\frac{\nu_1\rho_1}{\sqrt{h_1}} = \psi_1(M)$$

$$\frac{\nu_1}{h_1}\left(\rho_1^2 + \frac{1}{2}\right) = \psi_2(M) \tag{78}$$

$$\frac{\nu_1\rho_1}{h_1^{\frac{1}{2}}}\left(\rho_1^2 + 1 + \frac{1}{2}f\right) = \psi_3(M),$$

where M is the rate of evaporation, and ψ_1, ψ_2, and ψ_3 follow from the transfer equations for (76) and (77). Proceeding as before, we have

$$\frac{\rho_1^2(\rho_1^2 + 1 + \frac{1}{2}f)}{(\rho_1^2 + \frac{1}{2})^2} = \psi(M) \tag{79}$$

[12] In the case of a liquid evaporating through a straight tube and discharging into a region of pressure P, the pressures P and p_1 coincide up to the critical point, beyond which p_1 and the rate of discharge remain constant as P is lowered. This behavior is closely analogous to that encountered in the case of a gas discharging through an orifice from a region of high pressure to one of lower pressure.

Goodenough, "Principles of Thermodynamics", p. 146.

"Handbuch der Experimentalphysik", vol. 4, part 1, p. 387.

in place of (43); and in place of (64) and (65) we have

$$0 \leqq \psi(M) \leqq \frac{(1 + \frac{1}{2}f)^2}{1 + f}, \tag{80}$$

an expression which limits the range of M. The curve Fig. 7 now follows, where α is replaced by $\psi(M)$; and we have a horizontal tangent at the critical point, at which the isotropic gas is moving with its sound velocity. If, now, we assume from (61) and the physical nature of the problem that ρ_1 must be a one-valued function of M; then it follows that one of the two branches of the curve meeting at the critical point cannot exist physically. Since $\rho_1 = 0$, corresponding to the stationary saturated vapor, is possible, it follows that the right hand branch of the curve is not; hence we again obtain the portion of the curve lying between the origin and the critical point. We have thus obtained the above results with the arbitrary distributions (76) and (77) instead of the assumed distributions (10) and (13), these results now being subject to the assumption that ρ_1 is a one-valued function of M.

9. **Calculations, Tables, and Curves for Diatomic Gases in their Final Isotropic State.** The calculations indicated in Section 7 were carried out for diatomic gases ($f = 5$); and the results are given in Table III and in Figs. 8 and 9. The curves Fig. 9 were obtained by plotting $\frac{T_1}{T_0}$, $\frac{p_1}{p_0}$, and $\frac{\nu_1}{\nu_0}$ against $\frac{M}{M_0^+}$ for the same values of ρ, thus eliminating ρ. The comments made in Section 5 concerning the computations also apply here.

The value of ρ at the critical point is .457456 $\cdots$, and the data corresponding to this point are printed in heavy type in Tables II and III. In carrying out the computation of this critical data the required values of $\frac{2}{\sqrt{\pi}} \int_0^\rho \epsilon^{-x^2}\, dx$ and $\frac{2}{\sqrt{\pi}}\, \epsilon^{-\rho^2}$ were obtained from the tables of Burgess[8] by means of Lagrangean interpolation,[13] the expression for the error being used to hold the error below specified limits. Here again all figures tabulated are significant.[9]

Referring to Table II, we see that the only tabulated values of ρ

[13] Rutledge and Crout, "Tables and Methods of Extending Tables for Interpolation Without Differences", *Journal of Mathematics and Physics*, vol. 9, p. 166. Values of the derivatives required in the expression limiting the error were taken from Czuber, "Wahrscheinlichkeitsrechnung", vol. 1, p. 462.

[9] The critical value of ρ for $f = 5$ was found to lie between .4574567 and .4574568.

lying in the region corresponding to $1 < \alpha < \alpha_0$ are .30, .35, .40, and .45. Data corresponding to the stable region to the left of and including the critical point on Fig. 7 are tabulated above the double horizontal line in Table III, whereas data corresponding to the unstable region to the right of the critical point are tabulated below this line. Comparing column $\frac{M}{M_0^+}$ in Table I and column $\frac{p_1}{p_0}$ in Table III, and noting (71), we see that the region to the right of the critical point is unstable not only in the immediate vicinity of that point, as was proved in Section 8, but also throughout its entire extent. Since only the stable region can be obtained physically, this region alone has been included in the curves Figs. 8 and 9. These curves thus *stop* at the critical point, at which they have vertical tangents.[12]

Comparing Figs. 6 and 9 we see that the temperature of the isotropic gas lies between the two temperatures of the anisotropic gas. Similarly we see from Figs. 5 and 9 that the pressure of the isotropic gas is smaller than the pressure on the liquid surface unless the rate of evaporation is zero, in which case the two pressures are equal. This fact also follows from (57) for any value of f, for (61) requires that $\rho_1 \geqq 0$, the equality occurring when the rate of evaporation is zero.

In the tables and curves which follow, the denominators of the tabulated ratios are the constants of the stationary saturated vapor. Since the various physical tables usually give only the saturation vapor pressure as a function of temperature, it will be convenient if we give the various constants in terms of pressure and *absolute* temperature. Thus proceeding as with a monatomic gas, we have

$$\begin{aligned} &\text{Density} = m\nu_0 = \frac{\bar{m}p_0}{RT_0}, \\ &\text{Rate of emission} = M_0^+ = p_0\sqrt{\frac{\bar{m}}{2\pi RT_0}}, \\ &\text{Mean free path} = \lambda_0 = \frac{1}{\pi\nu_0 S^2\sqrt{2}} = \frac{kT_0}{\pi p_0 S^2\sqrt{2}}, \end{aligned} \tag{81}$$

where

$$\begin{aligned} &\bar{m} = Am = \text{molecular weight}, \\ &R = Ak = \text{gas constant} = 8.314 \cdot 10.^7 \text{ ergs/degree per mole}, \\ &A = \text{Avogadro's number}, \\ &S = \text{diameter of molecule}. \end{aligned} \tag{82}$$

Again proceeding as with a monatomic gas we note that $\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0}$, and *assume* that $\frac{\lambda_0}{\lambda} \approx \frac{\nu}{\nu_0}$; and thus obtain the following result for diatomic gases:

$$\lambda_0 \leqq \lambda \leqq \lambda_1 < 3.20\,\lambda_0\,. \tag{83}$$

From this we see that a distance equal to several, say five or ten, times λ_1 or times $3.20\,\lambda_0$ should be sufficient to allow a diatomic gas to become isotropic.[11] Conversely if the gas encounters barriers or condensing surfaces within such a distance of the liquid surface, caution must be used in applying the above results.

10. **Limiting State of the Anisotropic Gas and of the Isotropic Gas as $f \to \infty$.** The calculations indicated in Sections 4, 7, and 8 have been carried out in the special cases $f = 3$ (monatomic gases) and $f = 5$ (diatomic gases). As the molecules of the gas become more complex, f becomes larger; and the question arises as to what happens as $f \to \infty$. We have already seen in Section 4 that $\frac{T_L}{T_0}$, $\frac{\nu}{\nu_0}$, $\frac{M}{M_0^+}$, $\frac{p_e}{p_0}$, and $\frac{h_L}{h_0}$ are independent of f; and thus remain unaltered as f increases. Using bars[14] to denote the limiting values of the various constants as $f \to \infty$, it follows from (27) and (28) that if $\lim_{f\to\infty} \gamma f = \infty$, then

$$\bar{\sigma} = \frac{6}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2 - \tfrac{1}{2}\,,$$

or

$$\bar{\sigma} = \tfrac{1}{2}\,(3h_L' - 1)\,. \tag{84}$$

On the other hand if $\lim_{f\to\infty} \gamma f = 0$, then we should have

$$\bar{\sigma} = \sigma_\infty$$

in place of (84). It is interesting to note that both of these limiting values are independent of γ. If γf approaches some positive value as

[14] The one exception to this convention is $\bar{m}$, which was defined by (82). Since $\bar{m}$ is the molecular weight, there is no possibility of confusion.

$f \to \infty$, we should expect σ to approach a value lying somewhere between σ_∞ and that given by (84). In any case we have from (29)

$$\frac{\overline{T}_T}{T_0} = \frac{\bar{\sigma}}{h_L'} . \tag{85}$$

Turning to the isotropic gas, we see from (46) that

$$\operatorname*{Lim}_{f\to\infty} \frac{\alpha}{f} = \frac{\rho^2 h_L'}{2(\rho^2 + \frac{1}{2})^2} , \tag{86}$$

which with (49) gives

$$\bar{p}_1^2 = -\frac{1}{2} + \frac{1}{r} - \frac{1}{r}\sqrt{1-r} ,$$

or

$$\bar{p}_1^2 = -\frac{1}{2} + \frac{1}{1+\sqrt{1-r}} , \tag{87}$$

where

$$r = \frac{2\rho^2 h_L'}{(\rho^2 + \frac{1}{2})^2} . \tag{88}$$

From (57) it now follows that

$$\frac{\bar{p}_1}{p_0} = \left[\frac{1}{2(\bar{p}_1^2 + \frac{1}{2})}\right] \frac{p_s}{p_0} . \tag{89}$$

Noting that $\bar{p}_1^2$ is a function of ρ alone, we see from (51) that

$$\bar{h}_1' = \frac{1}{h_L'} ,$$

which expression together with (53) and (54) gives the rather unexpected result

$$\frac{\overline{T}_1}{T_0} = 1 . \tag{90}$$

It now follows from (58) that

$$\frac{\bar{\nu}_1}{\nu_0} = \frac{\bar{p}_1}{p_0} . \tag{91}$$

For the critical point we have

$$(\bar{\rho}_1)_{\text{crit.}} = \frac{1}{\sqrt{2}}, \tag{92}$$

due to (63); also it follows from (64), (86), and (88) that for the critical point

$$r = 1. \tag{93}$$

This relation determines the critical value of ρ. Knowing the critical values of ρ and $\bar{\rho}_1$, the other critical data now follow from (89), (90), and (91).

Noting (60) we see that the limiting value of the velocity of sound is

$$\bar{u}_s = \frac{1}{\sqrt{2\,\bar{h}_1 m}} = \sqrt{\frac{R\bar{T}_1}{\bar{m}}} = \sqrt{\frac{RT_0}{\bar{m}}} \tag{94}$$

due to (90). It is thus evident that $\bar{u}_s$ is equal to the velocity of sound in the stationary saturated vapor.

The calculations outlined in the last few paragraphs have been carried out for the stable region; and the results are given in Table IV and in Fig. 10. All figures tabulated in Table IV are significant[9] with the exception of the critical data, which are printed in heavy type. The figures tabulated for the critical data are probably significant.

For values of f greater than five we should expect the corresponding curves to lie between those of Fig. 9 and those of Fig. 10; thus in a sense we have a set of bands instead of a set of curves, which are applicable to any gas regardless of f. This is probably the best that can be hoped for by way of a single set of data applicable to all polyatomic gases.

11. **Modification of Classical Theory to Meet Quantum Conditions.** The theory in Sections 2–10 inclusive was developed entirely on the basis of classical kinetic theory, which is characterised by an equipartition of energy, and a constancy of specific heats. We shall now modify this theory to meet quantum conditions; hence we must examine the fundamental assumptions given in Section 2, modify these assumptions if they conflict with the above conditions, and trace the effect of such modifications through the theory which follows.

Referring to Section 2 we see that the only objectionable assumption is the relation (3), which depends upon the equipartition theorem. Since different molecules may have different degrees of excitation, we

see that f loses its meaning completely; however, the important fact that the coefficient of dN in (3) does not contain u, v, or w does *not* conflict with the quantum conditions.[15] We shall now *define* f as a function of temperature and pressure by the relation

$$\tfrac{1}{2} RTf(T, p) = U(T, p) , \tag{95}$$

where $U(T, p)$ is the total energy in ergs per mole of the gas when stationary and isotropic. Here the state corresponding to zero energy is taken to be one wherein the molecules have no translatory motion and no rotary motion, wherein the various internal motions are entirely unexcited, and wherein the molecules are all infinitely far removed from each other. It is now assumed that U is given as a function of temperature and pressure, it being evident that this data together with the molecular weight completely describes the gas for our purposes,[4] and replaces that data which we lost when we discarded the equipartition theorem. From (95) we see that the mean energy per molecule is $\tfrac{1}{2} kTf$, or $\frac{f}{4h}$; whence subtracting the energy of translation, and reasoning as with (3), we obtain the relation

$$dE_i = \frac{f(T_0, p_0) - 3}{4h_0} dN , \tag{96}$$

which replaces (3) and constitutes the only modification necessary in our original assumptions.

Proceeding as before, it follows that the only modification required in the equations in Sections 3, 4, and 6 is that f be replaced by $f(T_0, p_0)$ throughout. Noting (28), (29), (30), (31), and (34), we see that *the expressions for* $\frac{h_L}{h_0}$, $\frac{T_L}{T_0}$, $\frac{\nu}{\nu_0}$, $\frac{p_s}{p_0}$, *and* $\frac{M}{M_0^+}$ *contain neither* f *nor* γ; *hence the values of these quantities given in Table I and in Figs.* 1, 2, 3, 4, 5, *and* 6 *are good for all gases regardless of the complexity or the degree of excitation of the molecules.* Also, since in (27) f appears only in terms having γ as a factor, it follows that the data for $\frac{T_T}{T_0}$ corresponding to $\gamma = 0$ is good for all gases. Noting (36) we see that *the physical significance given to the two temperatures* T_L *and* T_T *in Section* 6 *remains unchanged.*

Turning to Section 7 we see that the required modifications consist in replacing f by $f(T_0, p_0)$ in those expressions which arise from the transfer equations for the anisotropic gas, and in replacing f by $f(T_1, p_1)$ in

[15] "Handbuch der Physik", vol. 10, p. 282.

those which arise from the transfer equations for the isotropic gas. We thus have the following modifications:

a) In equations (39), (41), and (42), f is replaced by $f(T_1, p_1)$ in the left hand members, and by $f(T_0, p_0)$ in the right hand members.

b) In equations (43), (48), and (49), f is replaced by $f(T_1, p_1)$.

c) In equations (44), (45), (46), and (50) f is replaced by $f(T_0, p_0)$. (97)

d) In the numerator of the right hand side of (51), f is replaced by $f(T_1, p_1)$; in the denominator, by $f(T_0, p_0)$.

It is now clear that the various mathematical processes carried out in Section 7 have required no modification whatsoever. In particular we see that γ drops out of (44), leaving (46); whence it again follows that α *and all of the following isotropic gas data are independent of* γ.

Turning to Section 8, we see that f must be replaced by $f(T_1, p_1)$ in (59). The velocity of sound in the isotropic gas is now given by the expression[16]

$$u_s = \sqrt{\left(\frac{dp}{dD}\right)_{\substack{\text{adiabatic} \\ T=T_1,\ p=p_1}}}, \tag{98}$$

where D is the density; hence we proceed to determine $\left(\frac{dp}{dD}\right)_{\text{ad.}}$. For an adiabatic process we have

$$\frac{\partial U}{\partial T}\,dT + \frac{\partial U}{\partial p}\,dp = -\,p dV\,, \tag{99}$$

V being the volume per mole. Noting that

$$V = \frac{\bar{m}}{D}\,, \qquad \frac{p}{D} = \frac{R}{\bar{m}}\,T\,, \tag{100}$$

we have at once

$$dV = -\,\frac{\bar{m}}{D^2}\,dD\,, \qquad \frac{R}{\bar{m}}\,dT = \frac{D dp - p dD}{D^2}\,.$$

[16] Rayleigh, "Theory of Sound", vol. 2, p. 33.

whence substituting in (99), and solving for $\frac{dp}{dD}$, it follows that

$$\left(\frac{dp}{dD}\right)_{\text{adiabatic}} = \frac{p}{D}\left[\frac{\frac{\partial U}{\partial T} + R}{\frac{\partial U}{\partial T} + \frac{RD}{\bar{m}}\frac{\partial U}{\partial p}}\right]. \tag{101}$$

Referring to (95), we see that

$$\frac{\partial U}{\partial T} = \frac{1}{2}R\left[f + T\frac{\partial f}{\partial T}\right]$$

$$\frac{\partial U}{\partial p} = \frac{1}{2}RT\frac{\partial f}{\partial p};$$

whence substituting in (101) and then in (98), noting (16) and (100), we have finally

$$u_s = \sqrt{\frac{1}{2h_1 m}\left[\frac{f + 2 + T\frac{\partial f}{\partial T}}{f + T\frac{\partial f}{\partial T} + p\frac{\partial f}{\partial p}}\right]_{\substack{T=T_1\\p=p_1}}}, \tag{102}$$

the subscript "1" merely indicating that u_s refers to the gas in its final isotropic state. This expression agrees with (60) only when f is constant. The relation (61) now becomes

$$\rho_1 = \frac{u_1}{u_s}\sqrt{\frac{1}{2}\left[\frac{f + 2 + T\frac{\partial f}{\partial T}}{f + T\frac{\partial f}{\partial T} + p\frac{\partial f}{\partial p}}\right]_{\substack{T=T_1\\p=p_1}}}. \tag{103}$$

In view of (43) and (97), we should expect the curve of α against ρ_1 to be of the same general shape as Fig. 7, though somewhat distorted, since f is now a function of T_1 and p_1, and hence of ρ_1 in (43). At the critical point $d\alpha/d\rho_1 = 0$; hence differentiating (43) and equating the derivative to zero, it follows that

$$1 - xf(T_1, p_1) + \frac{1}{2}f(T_1, p_1) + x\left[x + \frac{1}{2}\right]\left[\frac{\partial f(T_1, p_1)}{\partial T_1}\frac{dT_1}{dx} + \frac{\partial f(T_1, p_1)}{\partial p_1}\frac{dp_1}{dx}\right] = 0, \tag{104}$$

where $x = \rho_1^2$. But from (52), (53), (54), and (57) we have

$$T_1 = \frac{T_0}{h_1' h_L'} = \left[\frac{T_0(\rho^2 + \tfrac{1}{2})^2}{h_L' \rho^2}\right]\left[\frac{x}{(x + \tfrac{1}{2})^2}\right], \qquad p_1 = \left[\frac{p_e}{2}\right]\left[\frac{1}{x + \tfrac{1}{2}}\right]; \tag{105}$$

hence since h_L' and p_e are functions of ρ, and since $d\alpha/d\rho_1 = 0$ at the critical point, it follows that at the critical point

$$\frac{dT_1}{dx} = \frac{\partial T_1}{\partial x} + \frac{\partial T_1}{\partial \rho}\frac{d\rho}{d\alpha}\frac{d\alpha}{d\rho_1}\frac{d\rho_1}{dx} = \frac{\partial T_1}{\partial x}$$
$$\frac{dp_1}{dx} = \frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial \rho}\frac{d\rho}{d\alpha}\frac{d\alpha}{d\rho_1}\frac{d\rho_1}{dx} = \frac{\partial p_1}{\partial x}, \tag{106}$$

providing that $\frac{\partial T_1}{\partial \rho}$, $\frac{\partial p_1}{\partial \rho}$, $\frac{d\rho_1}{dx}$, and $\frac{d\rho}{d\alpha}$ all remain finite for the critical value $\rho = \rho_c$. It is immediately evident from (24), (28), (34), and (105) that $\frac{\partial T_1}{\partial \rho}$ and $\frac{\partial p_1}{\partial \rho}$ do not become infinite; also, since $x = \rho_1^2$, and since $\rho_1 > 0$ at the critical point, it follows that $\frac{d\rho_1}{dx}$ does not become infinite. It thus remains to show that $\frac{d\alpha}{d\rho}$ does not vanish at the critical point. It can readily be verified[17] that $\frac{d\alpha}{d\rho} > 0$ for $0 < \rho < 2$ if $f(T_0, p_0) \geqq 3$. We shall see later on in this section that problems can be solved taking account of quantum conditions, the only data required being sets of data computed on the basis of classical theory for different values of f. It thus follows that the possible values of ρ_c cannot lie outside of the range of values allowed on the basis of classical theory; hence noting (70) we see that $0 < \rho_c < .55$ if $F(T_1, p_1, T_0, p_0) \geqq 3$. Since $.55 < 2$, it now

[17] Noting (46) and (97) we see that α may be written

$$\alpha = \alpha_m + \frac{\rho^3 h_L'}{2(\rho^2 + \tfrac{1}{2})^2}[f(T_e, p_e) - 3],$$

where α_m is the value of α for a monatomic gas. Noting monatomic gas tables[2] and Table I, it may readily be verified that α_m and $\frac{\rho^3 h_L'}{(\rho^2 + \frac{1}{2})^2}$ are increasing functions of ρ in the interval $0 < \rho < 2$; hence the same is true of α if $f(T_e, p_e) \geqq 3$.

follows that $\frac{d\alpha}{d\rho}$ does not vanish at the critical point. We thus see that the relations (106) are established if $f(T_0, p_0) \geqq 3$ and $F(T_1, p_1, T_0, p_0) \geqq 3$. If, now, we assume that the potential energy (relative to that state for which $U = 0 \cdots$ see (95)) of the gas in the states corresponding to $\rho = 0$, ρ_s that is due to forces exerted by the molecules on each other is negligible compared with the internal energy[9] of the gas in these states; then it follows immediately from (95) and (112) that $f(T_0, p_0) \geqq 3$ and $F(T_1, p_1, T_0, p_0) \geqq 3$, for the three translational motions of the molecules are fully excited. The relations (106) are thus established.

From (105) and (106) we now have

$$\frac{dT_1}{dx} = \left[\frac{T_0(\rho^2 + \frac{1}{2})^2}{h_L' \rho^2}\right]\left[\frac{\frac{1}{2} - x}{(x + \frac{1}{2})^2}\right] = T_1\left[\frac{\frac{1}{2} - x}{x(x + \frac{1}{2})}\right]$$
$$\frac{dp_1}{dx} = \left[\frac{p_s}{2}\right]\left[\frac{-1}{(x + \frac{1}{2})^2}\right] = -\frac{p_1}{x + \frac{1}{2}}. \tag{107}$$

Placing (107) in (104), solving for x, and taking the square root, it follows that *at the critical point*[10]

$$\rho_1 = \sqrt{\frac{1}{2}\left[\frac{f + 2 + T\frac{\partial f}{\partial T}}{f + T\frac{\partial f}{\partial T} + p\frac{\partial f}{\partial p}}\right]_{\substack{T=T_1 \\ p=p_1}}}, \tag{108}$$

a relation which when substituted in (103) gives

$$u_1 = u_s. \tag{109}$$

We have thus again arrived at the result that *at the critical point the isotropic gas moves with its sound velocity*. If, now, we suppose that the curve of α against ρ_1 is of the same general shape as Fig. 7, it follows that we have a stable left branch and an unstable right branch meeting at the critical point,[11] at which point the drift velocity and the rate

[10] It may be noted that this result could not be obtained by placing the expression under the radical in (49) equal to zero, for f is now a function of T_1 and p_1, and hence of ρ_1; hence (49) no longer expresses ρ_1 as an explicit function of ρ.

[11] The reasoning by which we choose the left branch as stable and the right branch as unstable hinges on the assumption that ρ_1 is a one-valued function of M, this assumption being justified and used in the same manner as with (80). It is evident that the analysis of the stability of the two branches given in Section 8 no longer applies.[12]

of evaporation take their maximum values.[12] We have thus obtained for gases having total energies which are arbitrary functions of temperature and pressure the same results concerning the critical point and the velocity of sound which we obtained in Section 8. Proceeding as with (76) and (77), it follows immediately that these results are independent of the velocity distribution and the internal energy distribution assumed for the anisotropic gas at the liquid surface.

We have just seen how by means of the definition (95) we have been able to leave the equations of Section 7 unchanged except for the modifications (97), and have been able to obtain the same results concerning the critical point and the velocity of sound which we had obtained before under the assumption of constant specific heats. It is easy to see, however, that the amount of work involved in computing isotropic gas data from the equations of Section 7 has been increased enormously by the modifications (97), for the f appearing in (49) is now a function of T_1 and p_1, which are in turn given by (52), (53), (54), and (57). Also, since f is replaced by $f(T_0, p_0)$ in (46), it follows that a calculated set of isotropic gas data is good for only one value of T_0. We thus see that the net result of (97) is to change the situation from one wherein a reasonable amount of computation gave a set of isotropic gas data applicable to all gases of a certain type, and good for a wide range of liquid temperatures, to a situation wherein a tedious method of successive approximations yields a set of isotropic gas data which is good for a single gas and a single liquid temperature. Still further complicating the picture is the fact that (95) involves $U(T, p)$, the absolute value of which is difficult to determine. Our problem is now to improve this state of affairs.

We have seen that the isotropic gas data are independent of γ; hence our results are not altered if we place $\gamma = 0$ in (39), and in the equations which follow. Noting (16), (27), (47), (82), (95), and (97), we see that the last equation of (39) becomes

$$\frac{m\nu_1\rho_1}{2(h_1 m)^{\frac{1}{2}}}\left(\rho_1^2+\frac{5}{2}\right)-\frac{3m\nu_1\rho_1}{4(h_1 m)^{\frac{1}{2}}}+\frac{\nu_1\rho_1}{A\sqrt{h_1 m}}U(T_1, p_1)$$
$$=\frac{m\nu\rho}{2(h_L m)^{\frac{1}{2}}}\left(\rho^2+\frac{3}{2}+\sigma_m\right)-\frac{3\nu\rho}{4h_0\sqrt{h_L m}}+\frac{\nu\rho}{A\sqrt{h_L m}}U(T_0, p_0); \tag{110}$$

whence by virtue of the first equation of (39) it follows that the last term on the right hand side of (110) can be transposed and combined with the last term on the left hand side, leaving the term

$$\frac{\nu_1 \rho_1}{A\sqrt{h_1 m}} [U(T_1, p_1) - U(T_0, p_0)]. \tag{111}$$

We now place

$$F(T_1, p_1, T_0, p_0) = \frac{2}{R(T_0 - T_1)} [U(T_0, p_0) - U(T_1, p_1)]; \tag{112}$$

then (111) may be written

$$\frac{\nu_1 \rho_1}{\sqrt{h_1 m}} \left[\frac{F}{4h_1} - \frac{F}{4h_0}\right],$$

or, noting the first equation of (39),

$$\frac{m\nu_1 \rho_1}{4(h_1 m)^{\frac{1}{2}}} F - \frac{\nu\rho}{4h_0\sqrt{h_0 m}} F. \tag{113}$$

The second term of (113) may be transposed to the right hand side of (110); whence we see that (110) becomes identical with the expression obtained by placing $\gamma = 0$ and by replacing f by $F(T_1, p_1, T_0, p_0)$ in the third equation of (39). Proceeding as before, we thus obtain the same equations for the isotropic gas data, except that f is replaced by $F(T_1, p_1, T_0, p_0)$ throughout. We thus have obtained the following result: *If f be replaced by $F(T_1, p_1, T_0, p_0)$ in equations* (46), (49), *and* (51), *the resulting expressions satisfy the quantum conditions, are applicable to all gases, and constitute the required modification of the classical theory.* The equations (52), (53), (54), (56), (57), and (58) obviously hold without modification. It is evident from (112) that $F(T_1, p_1, T_0, p_0)$ is determined by the *difference* of the total energies corresponding to the states (T_0, p_0) and (T_1, p_1); hence *the given physical data on the gas need include only the operating range of temperature and pressure.* In fact we have no use for physical data outside of this range.

Suppose, now, that we have available sets of data such as Table III for any value of f whatsoever. Since $F(T_1, p_1, T_0, p_0)$ is merely a measure of the difference quotient of U against T over the operating range, we may easily guess an approximate value of $F(T_1, p_1, T_0, p_0)$; then choosing that set of data which has this value for f, we solve the problem as with classical theory. Taking the values of T_1, p_1, T_0, and p_0 so obtained, we now recompute $F(T_1, p_1, T_0, p_0)$, and repeat the process; thus we have a method of successive approximations by means of which we can solve problems involving gases having variable specific heats by solving problems on the basis of classical theory. It is interesting

to note that *the only data required by this method are the complete data required on the basis of classical theory*,[20] which consist of a set of tables such as Table III, or the corresponding curves, computed for different values of f. In view of the fact that as f varies from *five* to *infinity* the curves for the isotropic gas data shift only from Fig. 9 to Fig. 10, we should not expect the isotropic gas data to vary rapidly with f; hence we should expect this method of successive approximations to converge *very rapidly*.

So far we have tables of isotropic gas data for $f = 3$ (monatomic gases),[3] $f = 5$, and $f = \infty$, the last two being given in this paper; hence we have merely a skeleton of the data referred to in the preceding paragraph. It is expected that this data will be rounded out at some future time. In the mean time we may assume that the curves corresponding to values of f greater than five lie between those for $f = 5$ and those for $f = \infty$; thus we again arrive at the set of bands referred to in Section 10, these bands being applicable to all polyatomic gases.

We have already seen in connection with (46) that for any value of f the isotropic gas data can be computed directly, there being no need for making any calculations whatsoever on the anisotropic gas at the liquid surface. Values of h_L' and σ_m, which are required in such computations, are given in Table I, all figures tabulated being significant.[9]

Since we are dealing with the problem of evaporation, it is clear that

$$\begin{aligned} T_0 &< \text{critical temperature of gas},[21] \\ p_0 &< \text{critical pressure of gas};[21] \end{aligned} \tag{114}$$

in fact we have seen in Section 2 that we must keep well away from the critical point in order to justify the use of the ideal gas equation for the saturated vapor.[4] Also, on the basis of calculations already made it seems safe to say that

$$\begin{aligned} \tfrac{1}{3} T_0 < T_1 &\leqq T_0, \\ \tfrac{1}{3} p_0 < p_1 &\leqq p_0. \end{aligned} \tag{115}$$

These expressions serve to limit the range of temperature and pressure

[20] This is one reason why we obtained a complete picture of what happens on the basis of classical theory before considering quantum conditions.

[21] The word "critical" as used in (114) and in this one paragraph has that meaning which it had in Footnote 4, instead of that meaning assigned to it in Section 8.

in which we are interested, and over which we require physical data to be given on the gas.

Let us next consider the particular case of a gas whose specific heat at constant volume is a function only of temperature. We then have

$$C_p = C_v + R$$

$$u_s = \sqrt{\left(\frac{C_p}{C_v}\right)_{T=T_1} \frac{RT_1}{\bar{m}}} \tag{116}$$

$$U(T) = \int C_v dT + \text{constant},$$

where C_v and C_p are the specific heats at constant volume and at constant pressure, respectively, in *ergs per mole per degree absolute*. We also have, noting (112),

$$F(T_1, T_0) = \frac{2}{R(T_0 - T_1)} \int_{T_1}^{T_0} C_v dT, \tag{117}$$

or

$$F(T_1, T_0) = \frac{2}{R} C_{v,\xi}, \tag{118}$$

where $C_{v,\xi}$ is the average value of C_v over the temperature range T_1 to T_0. We thus see that $F(T_1, T_0)$ *is a measure of the average value of* C_v *over the operating range*, which fact gives us a very clear picture of $F(T_1, T_0)$, and hence a very simple means of making the first approximation in carrying out the method of successive approximations outlined above.

It may finally be pointed out that *if for a certain gas* $C_v = C_{v,\xi} =$ *constant over the interval* T_1 *to* T_0, *the problem may be solved on the basis of classical theory, the value of* f *being* $\frac{2}{R} C_v$. This follows immediately from (118) and the results just proved.

The total molecular energy flowing out per unit area of liquid surface per second may readily be obtained from (110); thus

$$E^+ - E^- = \frac{m\nu_1 \rho_1}{2(h_1 m)^{\frac{1}{2}}} \left(\rho_1^2 + \frac{5}{2}\right) - \frac{3\, m\nu_1 \rho_1}{4(h_1 m)^{\frac{1}{2}}} + \frac{\nu_1 \rho_1}{A\sqrt{h_1 m}} U(T_1, p_1). \tag{119}$$

If, now, we denote by L_0 the energy in ergs required to evaporate one gram of liquid (initially at temperature T_0) at a rate so slow that the drift energy of the evaporated gas is negligible in comparison with the

thermal energy, and if we denote by U_L the energy in ergs per mole of liquid at temperature T_0, then

$$\frac{U_L}{\bar{m}} + L_0 = \frac{1}{2h_0 m} + \frac{1}{Am}\, U(T_0, p_0), \tag{120}$$

a relation which may be obtained from (119) and the first equation of (39). It now follows at once from (119) and (120) that

$$E^+ - E^- - \frac{\nu_1 \rho_1 U_L}{A\sqrt{h_1 m}} = \frac{m\nu_1\rho_1}{2(h_1 m)^{3/2}}\,(\rho_1^2 + 1) + \frac{m\nu_1\rho_1}{\sqrt{h_1 m}}\, L_0 - \frac{\nu_1\rho_1}{2h_0\sqrt{h_1 m}} - \frac{\nu_1\rho_1}{A\sqrt{h_1 m}}\,[U(T_0, p_0) - U(T_1, p_1)], \tag{121}$$

or, noting (112) and the fact that the left hand side of (121) is the rate at which energy must be supplied per unit area of liquid surface to maintain the evaporation,

$$L = L_0 + \frac{1}{2}\,u_1^2 - \frac{R}{2\bar{m}}\,[T_0 - T_1][2 + F(T_1, p_1, T_0, p_0)], \tag{122}$$

where L is the energy in ergs per gram which must be supplied to maintain the evaporation. This expression can be used to determine the thermal gradient in the liquid.

List of Symbols

The following is a partial list of the symbols used in this paper.

T_0 = absolute temperature of liquid surface, or of stationary saturated vapor, in degrees Kelvin.

p_0 = pressure of stationary saturated vapor in dynes/cm.^{2}

ν_0 = molecular density of stationary saturated vapor in molecules/cm.^{3}

λ_0 = mean free path of stationary saturated vapor in cm.

p_s = pressure on liquid surface in dynes/cm.^{2}

M_0^+ = rate of emission of molecules from liquid surface in gm./sec.cm.^{2}

M = rate of evaporation in gm./sec. cm.^{2}

T_L = longitudinal temperature of anisotropic gas at liquid surface in degrees absolute, or Kelvin.

T_T = transverse temperature of anisotropic gas at liquid surface in degrees absolute, or Kelvin.

ν = molecular density of anisotropic gas at liquid surface in molecules/cm.^{3}

u_0 = drift velocity of anisotropic gas at liquid surface in cm./sec.

T_1 = temperature of gas in its final isotropic state in degrees absolute, or Kelvin.

ν_1 = molecular density of gas in its final isotropic state in molecules/cm.^{3}

p_1 = pressure of gas in its final isotropic state in dynes/cm.^{2}

u_1 = drift velocity of gas in its final isotropic state in cm./sec.

λ_1 = mean free path of gas in its final isotropic state in cm.

u_s = velocity of sound in gas in its final isotropic state in cm./sec.

ρ has no dimensions, and is used merely as a parameter in the adjoining tables and curves. See Section 8.

ρ_1 has no dimensions, and is discussed in Section 8.

γ has no dimensions, and is a measure of the balance between the internal and the external energy of the anisotropic gas at the liquid surface. See Sections 2 and 5.

f is the number of squared terms in the expression for the total energy of a molecule under the classical concept of a gas.[5]

m = mass of molecule in gm.

$\bar{m}$ = molecular weight.

k = Boltzman's constant = $1.371 \cdot 10.^{-16}$ ergs/degree.

R = gas constant = $8.314 \cdot 10.^7$ ergs/degree per mole.

A = Avogadro's number = $6.064 \cdot 10.^{23}$ molecules per mole.

ϵ = base of natural system of logarithms = 2.71828 $\cdots$.

π = ratio of circumference to diameter of a circle = 3.14159 $\cdots$.

A bar used with any letter except m indicates that that letter refers to the limiting state of the gas as $f \to \infty$.[14]

The word "critical" shall have that meaning assigned to it in Section 8, except where it is expressly stated otherwise.

Throughout this paper all units are C.G.S., and all temperatures are in degrees absolute, or Kelvin, except where it is expressly stated otherwise.

TABLE I

The Data Given in This Table Is Independent of f

ρ	$\frac{T_L}{T_0}$	$\frac{p}{p_0}$	$\frac{M}{M_0^*}$	$\frac{p_s}{p_0}$	$\frac{h_L}{h_0}$	$\rho^2+\frac{3}{2}+\sigma_m$
0	1.00000	1.00000	0	1.00000	1.000 000	2.500 000
.03	.97132	.96261	.10089	.93668	1.029 529	2.532 981
.06	.94338	.92759	.19163	.88137	1.060 018	2.568 889
.10	.90727	.88430	.29859	.81834	1.102 212	2.621 382
.15	.86391	.83513	.41275	.75394	1.157 531	2.694 540
.20	.82248	.79087	.50851	.70251	1.215 841	2.776 241
.25	.78291	.75096	.58887	.66143	1.277 282	2.866 667
.30	.74516	.71491	.65630	.62861	1.341 998	2.966 001
.35	.70915	.68230	.71288	.60240	1.410 134	3.074 433
.40	.67484	.65277	.76036	.58147	1.481 839	3.192 154
.45	.64215	.62597	.80019	.56477	1.557 262	3.319 359
.50	.61104	.60164	.83358	.55144	1.636 554	3.456 242
.60	.55330	.55938	.88500	.53235	1.807 342	3.759 831
.70	.50115	.52429	.92099	.52024	1.995 402	4.104 493
.80	.45415	.49504	.94609	.51259	2.201 908	4.491 770
.90	.41186	.47057	.96349	.50778	2.427 989	4.923 150
1.00	.37387	.45004	.97549	.50478	2.674 710	5.400 036
1.20	.30922	.41821	.98926	.50176	3.233 911	6.495 372
1.40	.25732	.39543	.99550	.50062	3.886 170	7.786 466
1.60	.21569	.37895	.99820	.50021	4.636 390	9.279 582
1.80	.18222	.36688	.99932	.50007	5.487 768	10.978 604
2.00	.15523	.35791	.99976	.50002	6.442 076	12.885 444
∞	0	$1/\pi$	1	$\frac{1}{2}$	∞	∞

TABLE II

The Data Given in This Table Was Computed for $f = 5$

ρ	$\frac{T_\gamma}{T_0}$ $\gamma=0$	$\frac{T_\gamma}{T_0}$ $\gamma=.5$	$\frac{T_\gamma}{T_0}$ $\gamma=1.$	σ $\gamma=0$	σ $\gamma=.5$	σ $\gamma=1.$	α
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0
.03	1.00248	1.00544	1.00722	1.03208	1.03513	1.03697	.01278
.06	1.00497	1.01081	1.01431	1.06529	1.07147	1.07518	.05151
.10	1.00832	1.01783	1.02354	1.11138	1.12187	1.12816	.14316
.15	1.01253	1.02641	1.03474	1.17204	1.18810	1.19774	.31747
.20	1.01678	1.03477	1.04557	1.23624	1.25812	1.27125	.54761
.25	1.02105	1.04292	1.05605	1.30417	1.33211	1.34887	.81856
.30	1.02534	1.05086	1.06617	1.37600	1.41025	1.43080	1.11382
.35	1.02964	1.05859	1.07595	1.45193	1.49275	1.51724	1.41768
.40	1.03395	1.06611	1.08541	1.53215	1.57981	1.60840	1.71680
.45	1.03827	1.07343	1.09453	1.61686	1.67162	1.70447	2.00102
*.457456...	1.03891	1.07451	1.09587	1.62989	1.68573	1.71924	2.04167
.50	1.04258	1.08056	1.10334	1.70624	1.76839	1.80568	2.26346
.60	1.05117	1.09422	1.12004	1.89983	1.97763	2.02430	2.70982
.70	1.05968	1.10712	1.13558	2.11449	2.20915	2.26504	3.04964
.80	1.06806	1.11928	1.15001	2.35177	2.46454	2.53221	3.29636
.90	1.07626	1.13071	1.16338	2.61315	2.74536	2.82468	3.46974
1.00	1.08424	1.14145	1.17577	2.90004	3.05304	3.14485	3.58878
1.20	1.09940	1.16090	1.19780	3.55537	3.75425	3.87357	3.72254
1.40	1.11330	1.17781	1.21651	4.32647	4.57716	4.72758	3.78055
1.60	1.12579	1.19238	1.23233	5.21958	5.52833	5.71358	3.80461
1.80	1.13682	1.20484	1.24565	6.23860	6.61187	6.83582	3.81417
2.00	1.14644	1.21543	1.25682	7.38544	7.82986	8.09651	3.81778
∞	$\left(2-\frac{\pi}{4}\right)\frac{3}{4}$	$\left(\frac{5}{2}-\frac{\pi}{4}\right)\frac{3}{5}$	$\left(3-\frac{\pi}{4}\right)$	∞	∞	∞	$\frac{12}{\pi}$

* This row contains data for the critical point, at which the isotropic gas moves with its sound velocity. See Sections 8 and 9.

TABLE III

The Data Given in this Table Was Computed for $f = 5$

ρ	$\frac{T_1}{T_0}$	$\frac{\rho_1}{\rho_0}$	$\frac{p_1}{p_0}$	ρ_1	$\frac{h_1}{h_0}$
0	1.00000	1.00000	1.00000	0	1.00000
.03	.98841	.94593	.93497	.030264	1.01173
.06	.97708	.89536	.87484	.061078	1.02345
.10	.96230	.83269	.80130	.10312	1.03917
.15	.94412	.76082	.71831	.15750	1.05919
.20	.92593	.69473	.64327	.21458	1.07999
.25	.90728	.63291	.57423	.27555	1.10220
.30	.88743	.57370	.50912	.34257	1.12685
.35	.86503	.51474	.44527	.42006	1.15603
.40	.83665	.45130	.37758	.51961	1.19524
.45	.78248	.36014	.28181	.70856	1.27799
*.457456···	.74486	.31470	.23441	.83666	1.34253
†.45	.69661	.27108	.18884	.99770	1.43553
†.40	.55165	.19392	.10698	1.48920	1.81274
†.35	.38275	.14823	.05673	2.19294	2.61270
†.30	.13185	.11167	.01472	4.56592	7.58460

* This row contains data for the critical point, at which the isotropic gas moves with its sound velocity. See Sections 8 and 9. The critical value of M/M_0^+ for $f = 5$ is .80555.

† The portion of this table lying below the double horizontal line corresponds to the unstable region beyond the critical point, which cannot exist physically. See Sections 8 and 9.

TABLE IV

The Data Given in this Table Was Computed for the Limiting State $f = \infty$

ρ	$\frac{\overline{T}_T}{\overline{T}_0}$	$\frac{\overline{T}_1}{\overline{T}_0}$	$\frac{\overline{p}_1}{\overline{p}_0}$, also $\frac{\overline{\rho}_1}{\overline{\rho}_0}$	$\overline{\rho}_1$	$\overline{\sigma}$ equation (84)
0	1.00000	1.00000	1.00000	0	1.00000
.03	1.01434	1.00000	.93495	.03044	1.04429
.06	1.02831	1.00000	.87469	.06180	1.09003
.10	1.04637	1.00000	.80062	.10521	1.15332
.15	1.06805	1.00000	.71608	.16260	1.23630
.20	1.08876	1.00000	.63800	.22484	1.32376
.25	1.10854	1.00000	.56348	.29480	1.41592
.30	1.12742	1.00000	.48819	.37924	1.51300
.35	1.14542	1.00000	.40039	.50226	1.61520
*.3773···	1.155	1.00000	.295	.707	1.673

* This row contains data for the critical point, at which the isotropic gas moves with its sound velocity. See Sections 8 and 10.

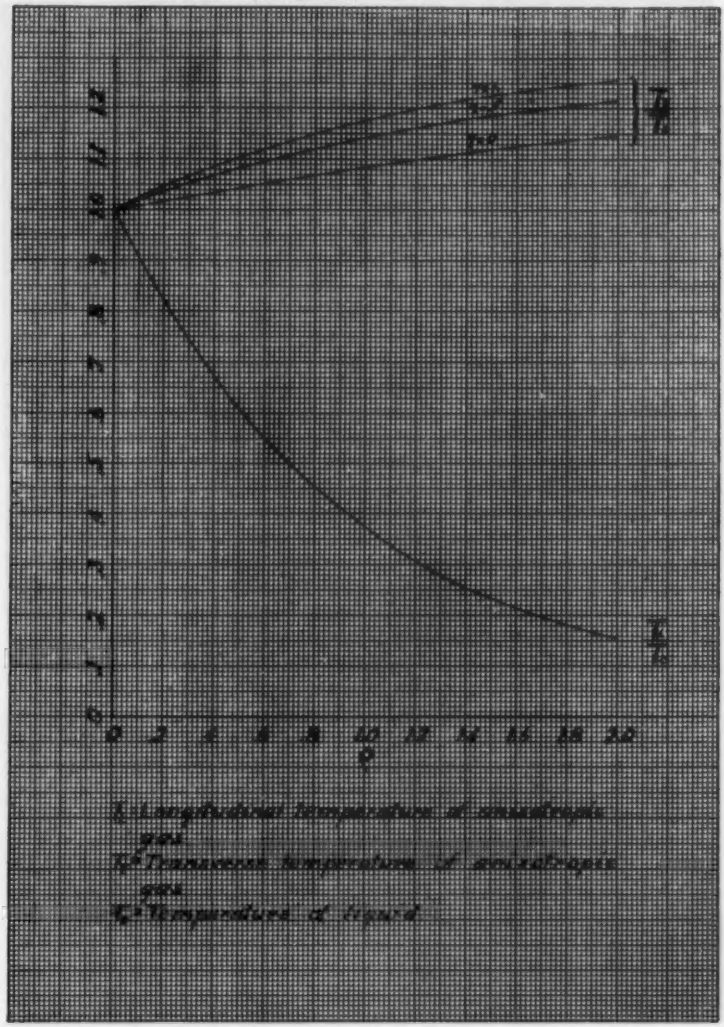

FIG. 1

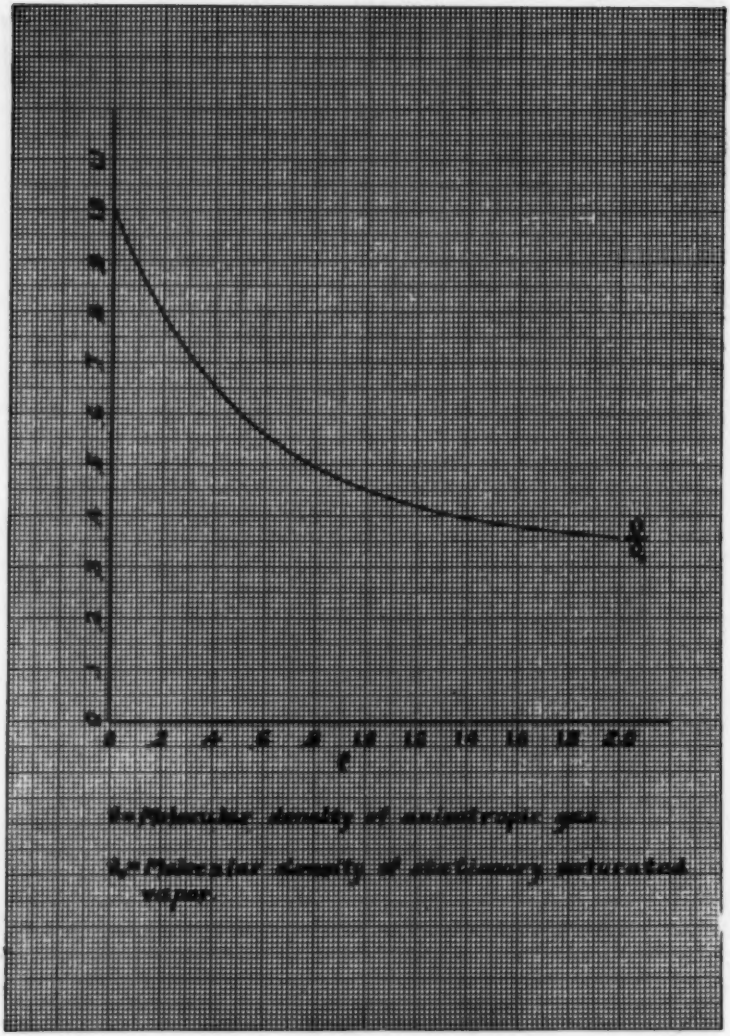

FIG. 2

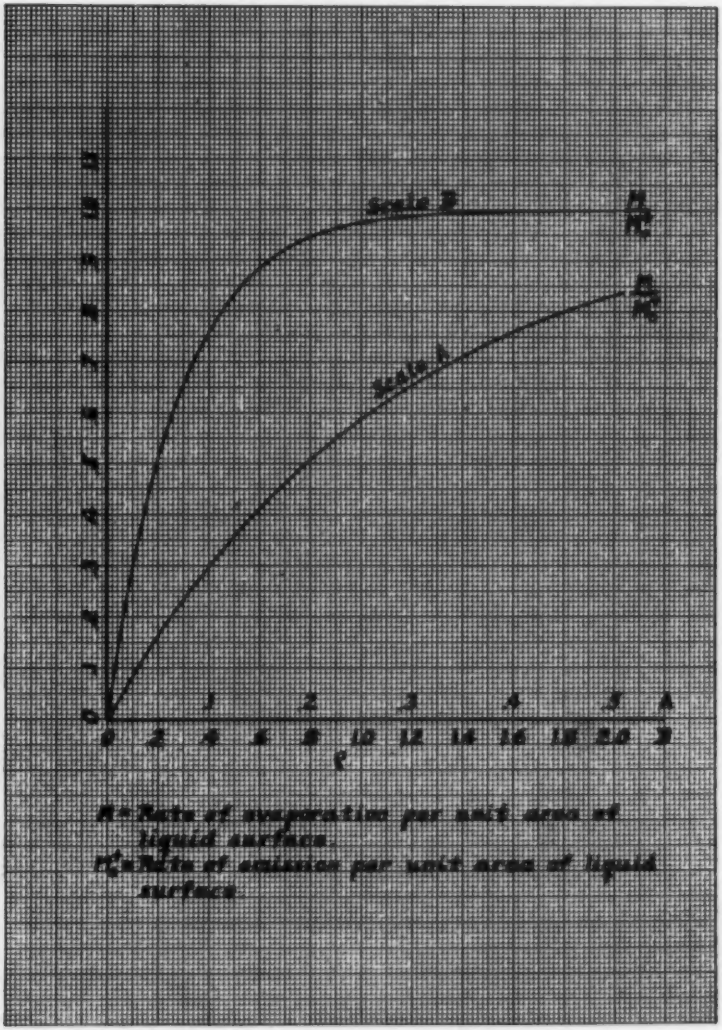
Scale B
Scale A
0 1 2 3 4 5 A
0 2 4 6 8 10 12 14 16 18 20 B
R = Rate of evaporation per unit area of liquid surface.
r = Rate of emission per unit area of liquid surface.

FIG. 3

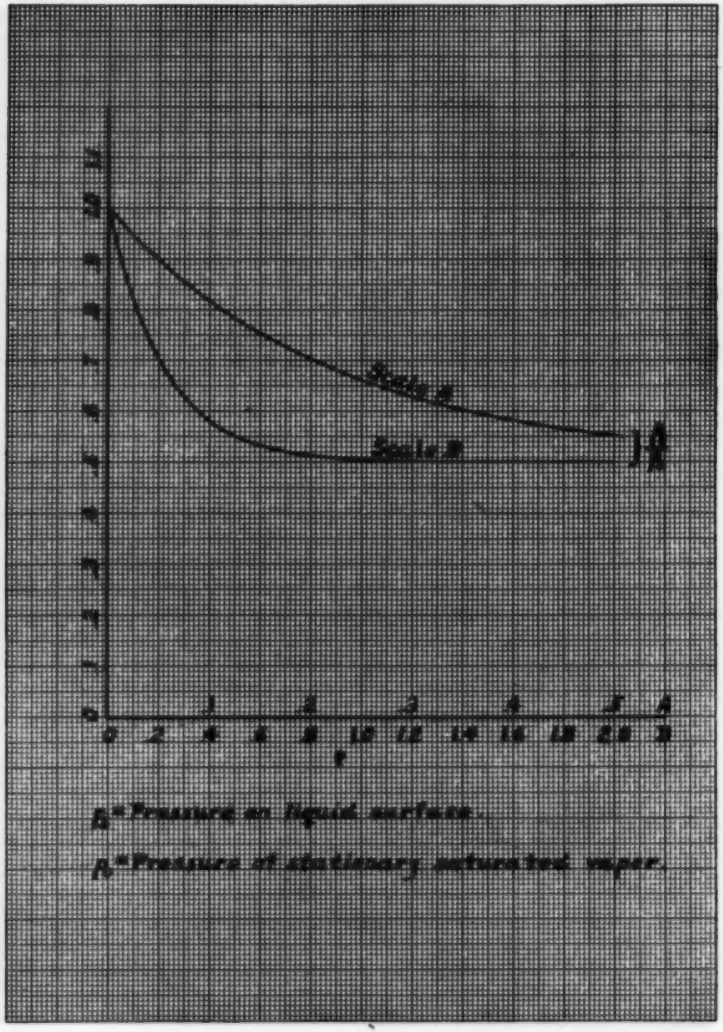
Scale A
Scale B
P = Pressure on liquid surface.
P_0 = Pressure of stationary saturated vapor.

FIG. 4

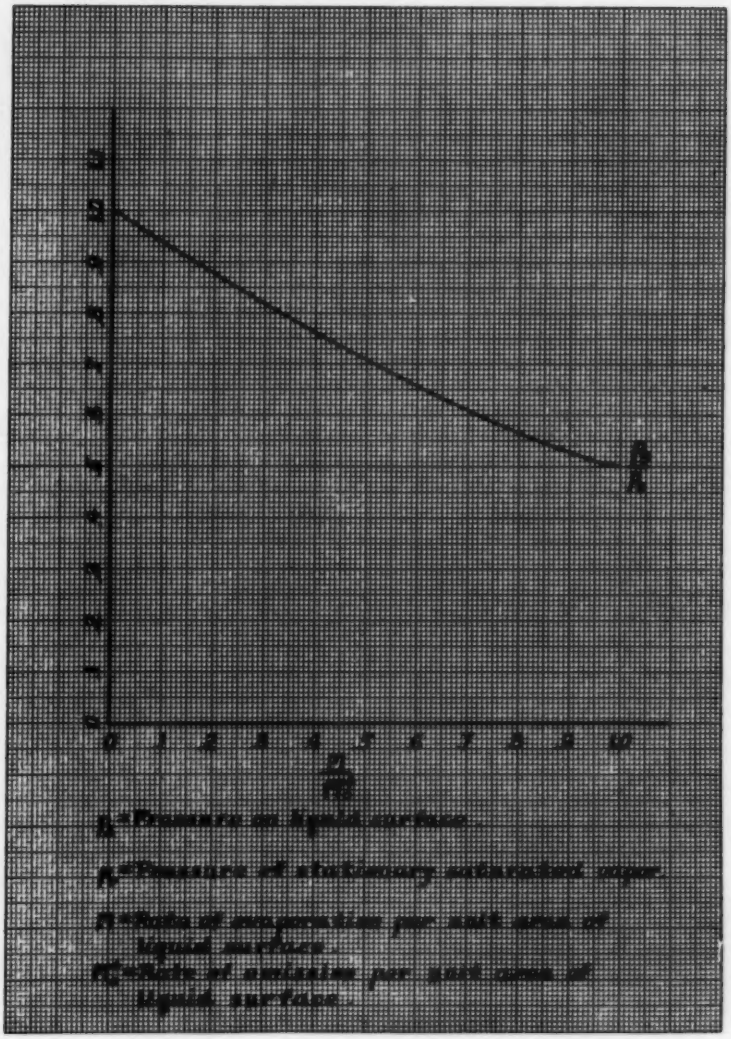

FIG. 5

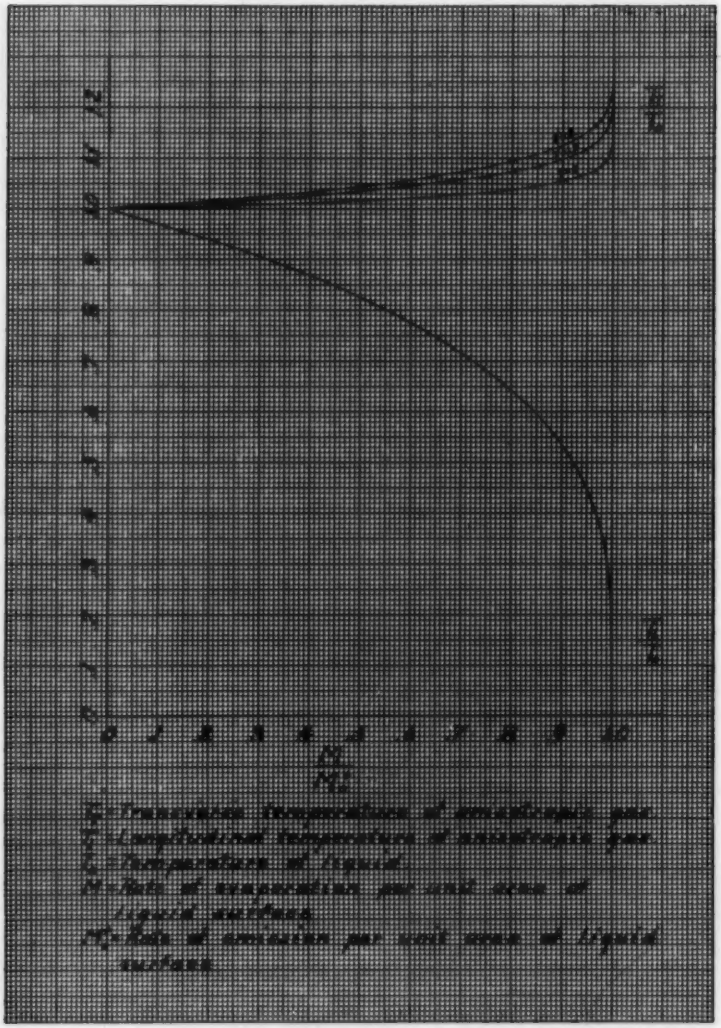

Fig. 6

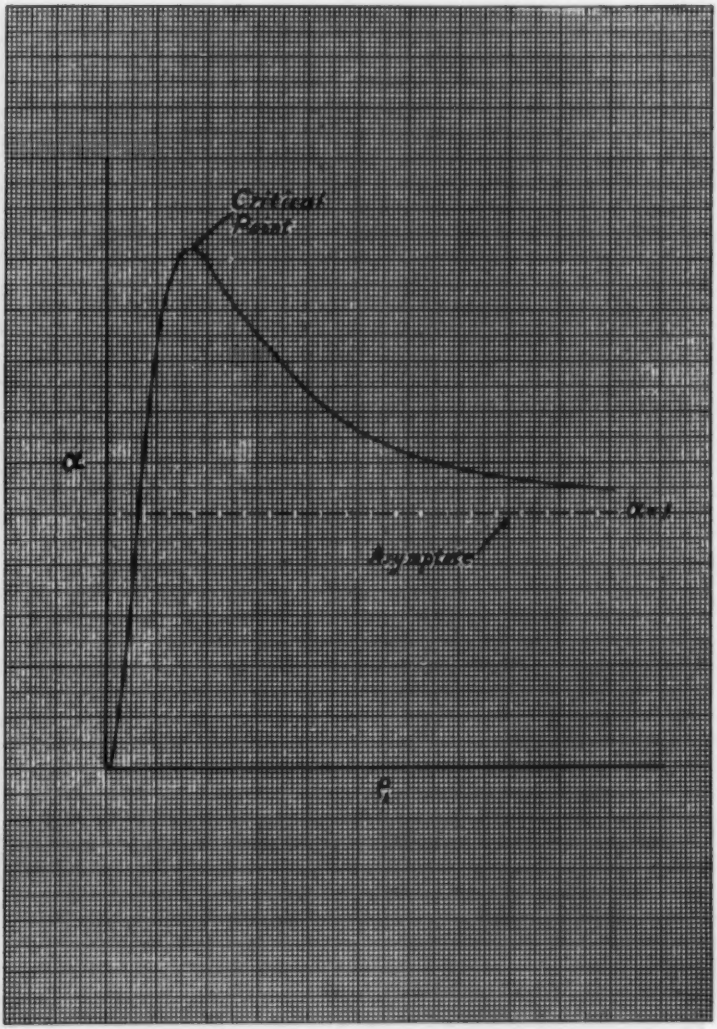

FIG. 7

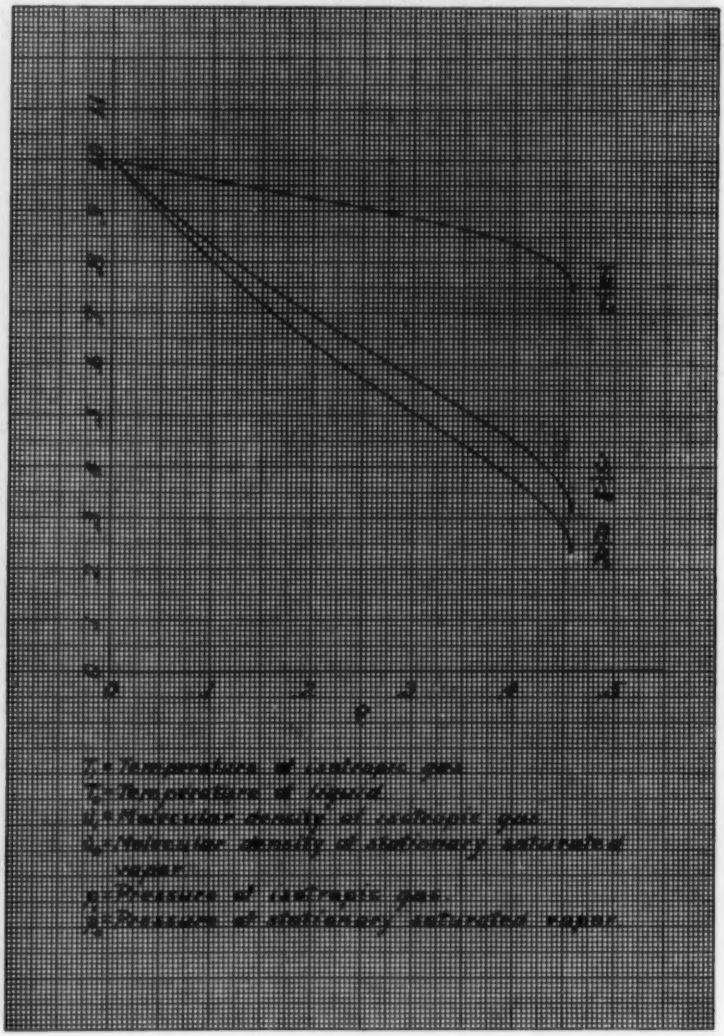

Fig. 8

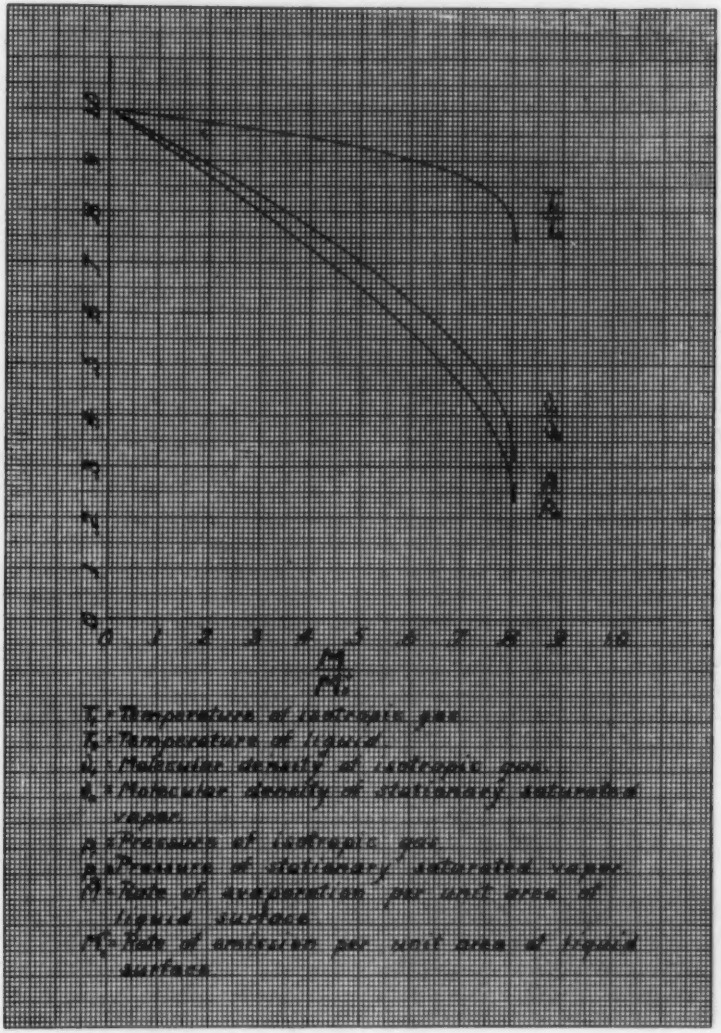
T_0 = Temperature of isotropic gas.
T_1 = Temperature of liquid.
n_0 = Molecular density of isotropic gas.
n_s = Molecular density of stationary saturated vapor.
p_0 = Pressure of isotropic gas.
p_s = Pressure of stationary saturated vapor.
Φ = Rate of evaporation per unit area of liquid surface.
Φ_0 = Rate of emission per unit area of liquid surface.

FIG. 9

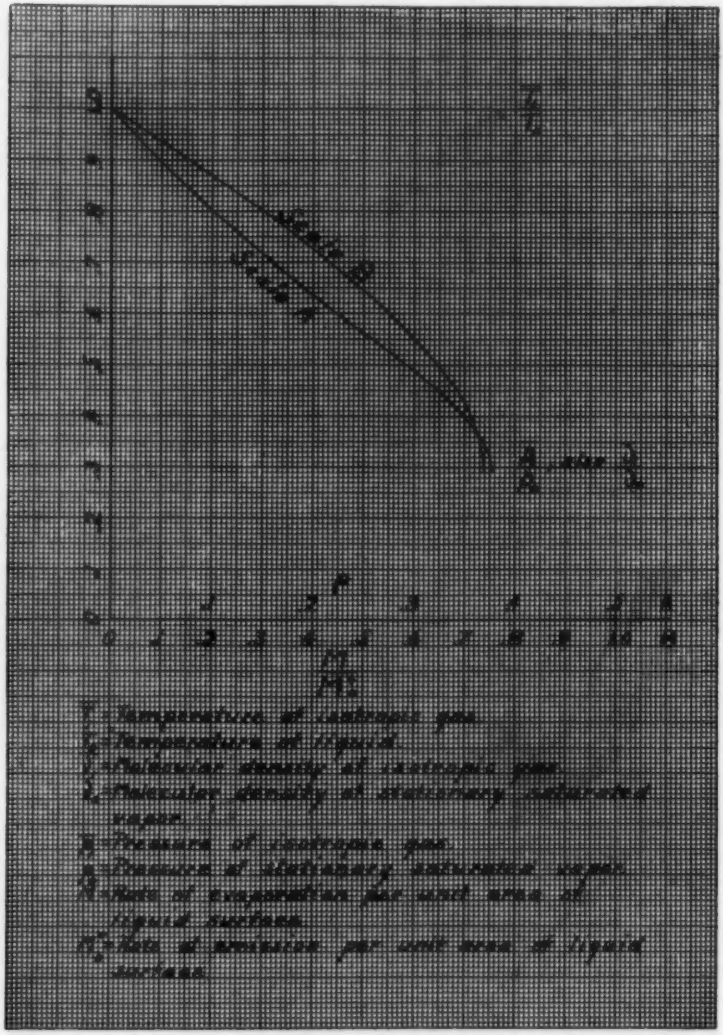

Fig. 10

THE TRIANGULATION OF SURFACES AND THE HEAWOOD COLOR FORMULA*

By I. N. Kagno†

Introduction. In this paper we prove a number of theorems on the triangulation of surfaces by graphs. We shall show that if a connected graph G can be mapped on a surface so as to triangulate the surface, then no matter how G is mapped, it triangulates the surface. We shall apply the special case where G is a complete N-point to the map coloring problem.

The Heawood color formula[1] gives an upper bound $[P_K]$, (where $[x]$ means the greatest integer in x), for the number of mutually touching regions which can be mapped on a surface of given characteristic, and hence for the number of colors sufficient to color such a map. Heawood did not show that that many colors actually are necessary. In fact Philip Franklin[2] has given an example where the number of colors indicated by the formula are not necessary. The question naturally arises as to whether the same may happen in other cases. Now for those surfaces of even characteristic K, for which the corresponding $[P_K] = P_K$, the maps consisting of P_K mutually touching regions are regular.[3] We shall show that for all such cases, an example in which the Heawood limit is attained fails to exist either for the orientable surface of characteristic K, or for the non-orientable surface of characteristic K.

Part I. Triangulation of Surfaces

Definitions. By a loop of type a we shall mean an arc with coincident endpoints. By a loop of type b we shall mean a circuit formed by two arcs. By a simple loop we shall mean a loop of type a or b.

By a complete N-point we mean the graph obtained by joining each

* Presented to the American Mathematical Society, April 10, 1936.

† Columbia University.

[1] P. J. Heawood, Quarterly Journal of Mathematics, Vol. 24 (1890), p. 332. Also Heffter, Math. Ann., Vol. 38 (1891), p. 477.

[2] Philip Franklin, "A Six Color Problem," Journal of Math. and Physics, Vol. 13 (1934), p. 363.

[3] See definition below.

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of N vertices to each of the others. A triangular 2-cell is a 2-cell bounded by a simple closed curve composed of three arcs and three vertices. A 2-cell whose boundary is a simple (i.e. non-singular) closed curve is called regular. A 2-cell whose boundary is a singular closed curve is called irregular. A graph is said to triangulate a surface, if when mapped on the surface it separates the surface into 2-cells, each of which is triangular. A graph G is said to separate a surface S, if $(S - G)$ is not connected. We shall denote a surface of characteristic K by the symbol $\mathfrak{S}_K$.

In the sequel, let G be a connected graph containing no simple loops, and no vertices of degree less than three.[4] Let a_i be the number of vertices of degree i in G, $(i = 3, 4, \cdots, a_0 - 1)$.

Lemma 1. Let G be mapped on a surface $\mathfrak{S}_r$ of characteristic r, in such a way as to triangulate the surface, if such a mapping is possible for the given graph. Then the number of 2-cells is given by

$$a_2 = \frac{\sum_{i=3}^{a_0-1} i a_i}{3}. \tag{1}$$

Proof: Since G contains no simple loops, the triangular 2-cells must all be regular, for; the boundary of an irregular 2-cell must contain as subset a simple loop. At each vertex of G there abut i 2-cells, where i is the degree of that vertex. Hence the total number of abutments at all vertices is

$$A = \sum_{i=3}^{a_0-1} i a_i. \tag{2}$$

But in this enumeration each 2-cell has been counted three times, once at each one of the vertices on its boundary. Hence the number of 2-cells is $A/3$, which is precisely that given in (1).

Definition. Let G separate a surface into 2-cells and α be one of these. Let P be an arbitrary, but fixed, vertex of $F(\alpha)$. Starting at P trace a closed path π around $F(\alpha)$, so moving that α is always to the right, and traversing the whole of $F(\alpha)$ terminate the path at P. By the number n, of sides of α, we mean the number of arcs in π, where in this enumeration each arc of G is counted as many times as π passes through it.

Lemma 2. Let G be mapped on an $\mathfrak{S}_t$, and suppose G separates $\mathfrak{S}_t$

[4] By the degree of a vertex we mean the number of arcs with which it is incident.

into 2-cells not all of which are triangular. Let α_2 be the number of 2-cells. Then

$$\alpha_2 < \frac{\sum i a_i}{3}. \tag{3}$$

Proof:* Let α_2^n be the number of 2-cells of n sides. Then $\alpha_2 = \Sigma \alpha_2^n$, $n \geqq 3$. At each vertex of G there abut a number of 2-cells, (where a given 2-cell α may abut more than once at the same vertex, if α is irregular). For each abutment of a 2-cell α with a vertex P, two sides of α are incident with P. If i is the degree of P, there are i abutments at P, and hence $2i$ sides are incident with P. Hence the total number of such incidences for all vertices is $B = 2\Sigma i a_i$. But in this enumeration each side has been counted twice, once at each of its ends, hence since the total number of sides is $\Sigma n \alpha_2^n$,

$$\sum n\alpha_2^n = \frac{B}{2} = \sum i a_i. \tag{4}$$

Since by hypothesis $n > 3$ for at least one 2-cell,

$$\sum n\alpha_2^n > 3\sum \alpha_2^n = 3\alpha_2$$

whence from (4) the lemma follows.

Lemma 3. If the graph G triangulates a surface $\mathfrak{S}_r$, and if G is mapped on a surface $\mathfrak{S}_t$ which it does not triangulate, then $r > t$.

Proof: On $\mathfrak{S}_r$, $\quad r = \alpha_0 - \alpha_1 + \alpha_2.$

On $\mathfrak{S}_t$, $\quad t = \alpha_0 - \alpha_1 + \alpha_2^*.$

By lemmas 1 and 2,

$$\alpha_2^* < \frac{\sum i a_i}{3} = \alpha_2,$$

therefore $r > t$.

Theorem 1. If a graph G when mapped on an $\mathfrak{S}_K$ does not separate $\mathfrak{S}_K$, then it can be mapped on a surface of greater characteristic.

Proof: Case I, $\mathfrak{S}_K$ is orientable; We trace a simple closed curve λ on $\mathfrak{S}_K$ which passes through a vertex P of G, but has no other point in common with G as follows; We start at P, then trace a closed path near $F(\mathfrak{S}_K - G)$, following an arbitrary, but fixed, direction until we first return to P. We then terminate the path at P. The path λ

* This simple proof was suggested to the author by Prof. Franklin in place of the original proof, which was cumbersome.

follows along a circuit of G, and can obviously be deformed into this circuit. λ does not separate $\mathfrak{S}_K$, since if it does, the circuit into which it is deformable separates $\mathfrak{S}_K$. Cut $\mathfrak{S}_K$ along λ. There results a surface $\mathfrak{T}^*$ with two holes bounded by λ_1 and λ_2 respectively, each λ_i corresponding to λ. λ_1 and λ_2 contain the vertices P_1 and P_2 respectively, each corresponding to P. If we span these holes by 2-cells, we obtain a closed surface $\mathfrak{S}_L$ on which is mapped the graph $G^* + \lambda_1 + \lambda_2$, where G can be formed from G^* by coalescing the vertices P_1 and P_2. Now since λ did not separate $\mathfrak{S}_K$, P_1 must be accessible to P_2 on $(\mathfrak{S}_L - G^*)$. Hence the vertices P_1 and P_2 can be brought arbitrarily close and made to coalesce. When this is done we shall have mapped G on $\mathfrak{S}_L$. By comparing characteristics we see that $L > K$.

Case II, $\mathfrak{S}_K$ is non-orientable; The proof is analogous to the proof of case I.

Definition. A surface $\mathfrak{S}$ is called maximal for a given graph if it is the surface of greatest characteristic upon which the graph can be mapped.

Corollary. If $\mathfrak{S}_K$ is maximal for G, then G separates $\mathfrak{S}_K$.

Theorem 2. A necessary and sufficient condition that $\mathfrak{S}_K$ be maximal for G is that G separate $\mathfrak{S}_K$ into 2-cells no matter how G is mapped on $\mathfrak{S}_K$.

Proof: A, If the surface is orientable: The condition is necessary, for suppose $\mathfrak{S}_K$ is maximal for G. The surface is then separated into a number of regions. If one of these regions is not a 2-cell, it is a 2-cell with handles affixed. We could then remove a handle and close the resulting holes, and G would then be mapped on a surface of greater characteristic, which gives a contradiction.

The condition is also sufficient, for if G could be mapped on an $\mathfrak{S}_{K+n}$, we can affix n handles to one of the regions into which $\mathfrak{S}_{K+n}$ is separated by G. When this is done we shall have mapped G on an $\mathfrak{S}_K$ in such a way that one of the regions into which it separates this surface is not a 2-cell, contrary to hypothesis.

B, If the surface is non-orientable. The proof is similar to that of case A, the words "cross cap"[5] replacing "handle" in the discussion.

Theorem 3. If G can be mapped on an $\mathfrak{S}_K$ in such a way as to triangulate $\mathfrak{S}_K$, then $\mathfrak{S}_K$ is maximal for G.

[5] A cross cap is a hemisphere with a circular hole, diametrically opposite points of the edge of the hole being identified. A non-orientable surface is representable by a sphere with p cross caps affixed, where $p = 2 - K$. (See v. Kerekjarto, "Vorlesungen uber Topologie," p. 151.)

Proof: By lemma 1, if α_2 is the number of 2-cells into which $\mathfrak{S}_K$ is separated, then

$$\alpha_2 = \frac{\sum ia_i}{3}. \tag{5}$$

Suppose $\mathfrak{S}_{K+\epsilon}$ is maximal for G. Then G separates $\mathfrak{S}_{K+\epsilon}$ into 2-cells no matter how it is mapped on the surface. Now these 2-cells are either triangular, or for some mapping of G on $\mathfrak{S}_{K+\epsilon}$ one of these is not triangular. If the latter were true, then by lemma 3 we would have $K + \epsilon < K$, which is absurd. If the former were true then by lemma 1, if α_2' is the number of 2-cells into which $\mathfrak{S}_{K+\epsilon}$ is separated, $\alpha_2' = \Sigma ia_i/3$ hence by (5) $\alpha_2' = \alpha_2$. Since $\alpha_0' = \alpha_0$, $\alpha_1' = \alpha_1$ we have $K + \epsilon = K$. That is, $\mathfrak{S}_K$ is maximal for G.

Definition. Let there be a map of q countries on a surface $\mathfrak{S}_K$. $\mathfrak{S}_K$ shall be called maximal for the map if it is the surface of greatest possible characteristic on which there exists a map of q countries having the same incidence relations among them as in the given map. Consider the graph formed by the boundaries of the countries in the given map. If each vertex of this graph is of degree three, the map is called regular.

Theorem 3′ (*dual of theorem* 3). If on $\mathfrak{S}_K$ there exists a regular map of q countries, each country touching at least two others, then $\mathfrak{S}_K$ is maximal for the map.

Theorem 4. If G, when mapped on an $\mathfrak{S}_K$, triangulates $\mathfrak{S}_K$, then G does so no matter how it is mapped on $\mathfrak{S}_K$.

Proof: $\mathfrak{S}_K$ is maximal for G. Hence no matter how G is remapped on $\mathfrak{S}_K$ the surface is separated into 2-cells. Suppose for some remapping of G, one of these 2-cells is not triangular. Let α_2 be the number of 2-cells in the original map, and α_2' be the number of 2-cells in the remapping. By lemma 3,

$$\alpha_2' < \frac{\sum ia_i}{3} = \alpha_2,$$

whereas by the Euler Formula, $\alpha_2' = \alpha_2$. This contradiction proves the theorem.

Theorem 4′ (*dual of theorem* 4). If there exists a regular map of q countries on a surface, then any other map of q countries on the same surface, having the same incidence relations among them as do the q countries in the original map, is also regular.

Corollary. If a complete N point triangulates a surface $\mathfrak{S}_K$, it does so no matter how it is mapped on the surface.

The map of a complete N point on a surface which it triangulates is thus essentially unique.[6]

Corollary: (*dual of above corollary*). If a given map of N mutually touching countries on $\mathfrak{S}_K$ is regular, then any map of N mutually touching countries on the same surface is regular.

Part II. The Heawood Formula

Let $[P_K] = [\frac{1}{2}(7 + \sqrt{49 - 24K})]$, where $[x]$ means the greatest integer in x. Heawood has shown that $[P_K]$ is an upper bound to the number of mutually touching countries which can be mapped on a surface of characteristic K. Whenever $[P_K] = P_K$ the map of $[P_K]$ mutually touching countries, if it exists on $\mathfrak{S}_K$, is regular.[7] Heffter's calculations for such maps may be summarised in the theorems 5 and 5′.

Definition. Let $\mathfrak{S}_K$ be a surface of characteristic K. By the genus of $\mathfrak{S}_K$ we mean the integer $p = (2 - K)/2$ when $\mathfrak{S}_K$ is orientable, or the integer $q = 2 - K$ when $\mathfrak{S}_K$ is non-orientable. In the former case we denote $\mathfrak{S}_K$ by the symbol S_p, in the latter case by the symbol S_q.

Theorem 5. A necessary condition that a regular map of N mutually touching regions exists on a surface of genus p, is that N and p satisfy the relations

$$(1)\qquad p = \frac{z^2 - 1}{48}, \qquad N = \frac{7 \pm z}{2}, \qquad \begin{cases} z = 24y \pm 1,\ 24y \pm 7 \\ y = 0, 1, \cdots \end{cases}$$

when the surface is orientable, or

$$(2)\qquad p = \frac{z^2 - 1}{24}, \qquad N = \frac{7 + z}{2}, \qquad \begin{cases} z = 12y \pm 1,\ 12y \pm 5 \\ y = 0, 1, \cdots \end{cases}$$

when the surface is non-orientable.

[6] For, let H be a complete N-point which triangulates $\mathfrak{S}_n$. Map H on another surface $\mathfrak{S}_n'$ of the same characteristic. H also triangulates $\mathfrak{S}_n'$. Then there is a 1-1 continuous correspondence between the points of $\mathfrak{S}_n$ and those of $\mathfrak{S}_n'$ so that if α is a 2-cell of $\mathfrak{S}_n$ corresponding to α' on $\mathfrak{S}_n'$ then $F(\alpha)$ corresponds to $F(\alpha')$. In other words, if H is mapped on $\mathfrak{S}_n$ and if the vertices of H are numbered $1, 2, \cdots, n$, then any other map of H on $\mathfrak{S}_n$ is equivalent to the original map with the numbers assigned to the vertices permuted.

[7] Heffter, loc. cit.

Theorem 5' (dual of theorem 5). A necessary condition that a complete N-point triangulate a surface of genus p is that N and p satisfy relations (1) when the surface is orientable and (2) when the surface is non-orientable.

A table of values, up to $K = -68$, is given below.

	Orientable		Non-orientable	
K	p	N	p	N
2	0	3, 4		
1			1	6
0	1	7	2	7
−1			5	9
−4			7	10
−10	6	12	12	12
−13			15	13
−20	11	15	22	15
−24	13	16	26	16
−33			35	18
−38	20	19	40	19
−49			51	21
−55			57	22
−68	35	24	70	24

Examining this table we see that the values of N which do not appear are 5, 8, 11, 14, 17, $\cdots$ that is,

A necessary condition that a complete N point triangulate a surface, (or that a map of N mutually touching regions is regular) is that $N \neq 5 + 3m$, $(m = 1, 2, \cdots)$.

A complete N-point will triangulate a surface if and only if its dual map is regular. A necessary condition for such maps is $[P_K] = P_K$. We now show that this condition is not sufficient.

Theorem 6. The upper bound $[P_K]$ given by Heawood and Heffter for the number of mutually touching countries which can be mapped on a surface of characteristic K, cannot be attained for one of the surfaces; (a) the orientable surface of characteristic K, (b) the non-orientable surface of characteristic K, when K is even and $[P_K] = P_K$.

Proof: For every even value of K, there are both orientable and non-orientable surfaces of characteristic K. Suppose K' is an even value of K, such that a complete N-point exists on $\mathfrak{S}_{K'}$, triangulating the surface. (The value of N can be computed from theorem 5.) By the corollary to theorem 4, this map of the complete N-point on $\mathfrak{S}_{K'}$ is

essentially unique. That is, the triangular 2-cells into which the surface is separated, have an essentially unique incidence relationship among them. These 2-cells thus piece together into an essentially unique surface which is either orientable or non-orientable. If $\mathfrak{S}_{K'}$ is orientable, then for the non-orientable surface of characteristic K' there does not exist a map of a complete N-point which triangulates the surface. Under duality there does not exist a map of N mutually touching countries which is regular, on the surface. If $\mathfrak{S}_{K'}$ is non-orientable, then for the orientable surface of characteristic K' there does not exist a map of a complete N-point triangulating the surface. Under duality there then does not exist a map of N mutually touching countries, which is regular, on the surface.

Example. $K = 0$. The complete 7-point triangulates a torus, i.e., orientable surface of characteristic zero. The dual map consists of seven mutually touching hexagons. Therefore the map of seven mutually touching hexagons cannot exist on a non-orientable surface of characteristic zero. This was first shown by Philip Franklin.[8]

[8] P. Franklin, loc. cit.

A GEOMETRICAL INTERPRETATION OF THE METHOD OF SYMMETRICAL COMPONENTS

By B. Hoffmann, Winston E. Kock, and M. H. L. Pryce[1]

Introduction. The method of symmetrical components is a process for finding the distribution of currents and voltages in an unbalanced three-phase electrical system. By resolving the unbalanced components of the known currents, voltages, and impedances into three sets of symmetrical components, the negative sequence, the positive sequence, and the zero sequence elements, the problem is reduced to one of finding the distribution of currents and voltages in a balanced system, and the solution is thereby greatly simplified.

Kron has recently shown[2] that the three sequence operations may be reduced to one matrix operation. It is the purpose of this paper to consider more in detail the characteristics of such matrix operations and to discuss their geometrical representation.

General. The voltage equations of a three phase generating system (transformer or alternator) may be written as

$$
\begin{aligned}
e_a &= Z_{aa} i^a + Z_{ab} i^b + Z_{ac} i^c, \\
e_b &= Z_{ba} i^a + Z_{bb} i^b + Z_{bc} i^c, \\
e_c &= Z_{ca} i^a + Z_{cb} i^b + Z_{cc} i^c,
\end{aligned} \tag{1}
$$

where the three windings a, b, c, possess impedances Z_{aa}, Z_{bb}, and Z_{cc}, respectively, and mutual reactances Z_{ab}, Z_{ac}, etc. The currents are written with superscripts to show the tensor character of the equations. In general all quantities will be complex.

The standard method of solution is to operate with the three sequence operators $S^0 = (1, 1, 1)$, $S^1 = (1, a^2, a)$, and $S^2 = (1, a, a^2)$, where 1, a, and a^2 are the three cube roots of unity. However it is to be observed that the equation (1) can be written in the matrix form

$$e = Zi$$

[1] The Institute for Advanced Study, Princeton, N. J.

[2] G. Kron, Gen. Elect. Rev., *38*, 242 (1935).

and that the sequence operations are equivalent to the matrix transformation

$$e' = S^{-1}ZSi',$$

that is to

$$e' = Z'i',$$

where

$$Z' = S^{-1}ZS,$$

and

$$S = \frac{1}{\sqrt{3}}\begin{pmatrix} 1, & 1, & 1 \\ 1, & a^2, & a \\ 1, & a, & a^2 \end{pmatrix}. \tag{2}$$

S is unitary since $SS^* = 1$, the unit matrix. (S^* is the transposed conjugate of S.)

The most important case arises when the generator impedances are balanced, i.e. when the impedance matrix has the special form given by $Z_{aa} = Z_{bb} = Z_{cc}$, and $Z_{ac} = Z_{ab} = Z_{ca}$, etc. A great simplification then results because of the property of the matrix S of transforming any matrix of the form

$$A = \begin{pmatrix} p, & q, & q \\ q, & p, & q \\ q, & q, & p \end{pmatrix}$$

into a diagonal matrix of the type

$$A' = \begin{pmatrix} x, & 0, & 0 \\ 0, & y, & 0 \\ 0, & 0, & y \end{pmatrix},$$

where

$$A' = S^{-1}AS.$$

When the above condition holds, the impedance matrix Z is of the form A and the method of symmetrical components is seen to be equivalent to a principal axis transformation

$$Z' = S^{-1}\,Z\,S,$$

$$Z' = \begin{pmatrix} Z_{aa} + 2Z_{ac}, & 0 & , & 0 \\ 0 & , Z_{aa} - Z_{ac}, & & 0 \\ 0 & , & 0 & , Z_{aa} - Z_{ac} \end{pmatrix}.$$

The geometrical significance of this principal axis transformation becomes evident when we observe that the matrix

$$A = \begin{pmatrix} p, & q, & q \\ q, & p, & q \\ q, & q, & p \end{pmatrix}$$

is a linear combination of the unit matrix and the matrix

$$B = \begin{pmatrix} 1, & 1, & 1 \\ 1, & 1, & 1 \\ 1, & 1, & 1 \end{pmatrix};$$

$$A = (p - q)\, 1 + q\, B.$$

The transformation that brings B to diagonal form will also bring A to diagonal form. B itself has a simple geometrical interpretation, for when it operates on any vector $x = (x^1, x^2, x^3)$ it changes it into a multiple of the vector $x_1' = (1, 1, 1)/\sqrt{3}$:

$$Bx = (x^1 + x^2 + x^3)\, \sqrt{3}\, x_1'.$$

In particular, any vector orthogonal to x_1' is reduced to zero by B, since for such a vector $x^1 + x^2 + x^3 = 0$. The principal axes of B are therefore the line parallel to x_1', and any two directions orthogonal to each other and to x_1'. Orthogonality of two vectors x, y is here understood in the unitary sense

$$(x, \bar{y}) = 0.$$

To preserve symmetry with respect to the three original axes we want the two new axes, which lie in the plane $x^1 + x^2 + x^3 = 0$, to make equal angles with the projections of the three axes on this plane. These projections are three lines making angles of 120° with each other, and it is easily proved that the vectors $(1, a, a^2)$, $(1, a^2, a)$ do indeed make equal angles with them. Further, they are orthogonal since

$$\begin{aligned} 1.\bar{1} + a.\bar{a}^2 + a^2.\bar{a} &= 1.1 + a.a + a^2.a^2 \\ &= 1 + a^2 + a \\ &= 0. \end{aligned}$$

The new axes must therefore be given by the unit vectors

$$x_1' = \frac{1}{\sqrt{3}}(1, 1, 1)$$

$$x_2' = \frac{1}{\sqrt{3}}(1, a^2, a)$$

$$x_3' = \frac{1}{\sqrt{3}}(1, a, a^2).$$

The factor $1/\sqrt{3}$ is required in order to make these vectors unit vectors in the sense

$$(x, \bar{x}) = 1.$$

The transformation of $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$ into x_1', x_2', x_3' is clearly given by the matrix equation

$$x' = Sx,$$

with S given by (2).

Now we have in particular

$$Bx_1' = 3x_1',$$
$$Bx_2' = 0,$$
$$Bx_3' = 0,$$

which can be written as

$$S^{-1}BS\, x_1 = 3x_1,$$
$$S^{-1}BS\, x_2 = 0,$$
$$S^{-1}BS\, x_3 = 0.$$

The matrix $S^{-1}BS$ is therefore

$$S^{-1}BS = \begin{pmatrix} 3, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}$$

and the matrix $S^{-1}AS$ is accordingly

$$S^{-1}AS = (p - q)\,1 + q\,S^{-1}BS$$
$$= \begin{pmatrix} p + 2q, & 0 & , & 0 \\ 0 & , & p - q, & 0 \\ 0 & , & 0 & , & p - q \end{pmatrix}.$$

AN EXTENSION OF GIBBS' VECTOR ANALYSIS TO *N*-SPACE

By I. M. Hostetter[1]

1. The systems of vector analysis for 4-space as developed by Sommerfeld,[2] Wilson and Lewis,[3] Weatherburn,[4] and others, have grown out of Minkowski's conception of a space-time vector. However valuable they may be in the treatment of the theory of relativity they do not readily admit of extension to n dimensions. In treating the four-dimensional problem Minkowski[5] introduced three different types of vectors, namely, directed line elements, directed surface elements, and directed volume elements, which were designated by him as vectors of the first, second, and third kind, respectively. Sommerfeld called vectors of the first and third type *four-vectors* and those of the second type *six-vectors*, vectors of the first type being represented as the set of four numbers designating the lengths of the projections of the line element upon the four axes, those of the second type being represented by the set of six numbers measuring the areas of the projections of the surface element upon the six coordinate planes, and those of the third type being represented by the set of four numbers measuring the volumes of the projections of the volume element upon the four coordinate planoids. Wilson and Lewis classified the three types as *one-vectors*, *two-vectors*, and *three-vectors*, in accordance with the dimensionality of the space occupied by each. Whatever the point of view, the combination and manipulation of these different types of vectors is at least confusing. The difficulties incident to the extension to a greater number of dimensions is obvious.

From the point of view which regards a vector as an entity in itself, distinguished by the properties magnitude and direction, and not primarily as a representation of a set of numbers, exception might be taken to the classification of the surface elements in 4-space as vectors. A surface element determines a definite direction—the normal to the

[1] Institute for Advanced Study.
[2] Ann. der Physic., Bd. 32, S. 749 (1910); Bd. 33, S. 649 (1910).
[3] Proc. Am. Acad., Vol. 48, p. 389.
[4] Quart. Jour., Vol. 48, p. 39; also Mess. Math., Vol. 49, p. 155; Vol. 50, p. 49.
[5] Gott. Nach. (1908), S. 53; Math. Ann., Bd. 68, S. 472 (1910).

surface—only in one case, that of three dimensions. In 2-space it indicates no direction whatever, and in 4-space not a single direction, but ∞' directions normal to the surface element. The three-dimensional element in 4-space does determine a vector normal to the element, but it would appear to be simpler to resolve this normal along the four axes than to project the element on the four coordinate planoids.

2. **Inner and outer products defined.** We recognize with Gibbs two functions of the indeterminate product **AB** of two vectors, namely the inner and outer products. The inner or scalar product is defined by

$$\mathbf{A}\cdot\mathbf{B} = AB\cos(\mathbf{A}, \mathbf{B}),$$

where A and B are the magnitudes of the vectors **A** and **B** and $(\mathbf{A}, \mathbf{B})$ is the euclidean angle between the two vectors. Discarding the idea of six-vectors, the outer product $\mathbf{A} \times \mathbf{B}$ for 4-space is no longer a vector whose magnitude is the area of the parallelogram determined by **A** and **B** as defined by the above writers, but rather a two-dimensional quantity—a dyadic—which can be reduced to a form wherein all of its antecedents and consequents are normal to the plane of **A** and **B**. Further, if we wish the product to conform to the requirement of 3-space that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, the dyadic must be antisymmetric. We can therefore express the product $\mathbf{A} \times \mathbf{B}$ in terms of two vectors normal to **A** and **B**.

$$\mathbf{A} \times \mathbf{B} = \mathbf{lm} - \mathbf{ml}.$$

If we further specify that

$$AB\sin(\mathbf{A}, \mathbf{B}) = lm\sin(\mathbf{l}, \mathbf{m})$$

the definition is complete. If we select the vector **l** so that it is orthogonal to **m**, then we can write the product in the convenient form:

$$\mathbf{A} \times \mathbf{B} = e(\mathbf{i}_3\mathbf{i}_4 - \mathbf{i}_4\mathbf{i}_3)AB\sin(\mathbf{A}, \mathbf{B}),$$

where $\mathbf{i}_3$ and $\mathbf{i}_4$ are mutually orthogonal unit vectors normal to **A** and **B**, and e has the value $+1$ or -1 depending upon the order of selection of the vectors $\mathbf{i}_3$ and $\mathbf{i}_4$. The value of e will later be defined analytically. The above definition of the outer product admits of immediate generalization to n-space.

We will adopt the notation of the tensor analysis, where a repeated index indicates summation on that index. $\epsilon_{i_1 \cdots i_n}$ has the value $+1$ or -1 depending upon whether the number of inversions from the

order 1, 2, $\cdots$, n is even or odd and vanishes for any repeated values of these indices, and δ^i_j has the value $+1$ or -1 according as $i = j$ or $i \neq j$. It is to be noted, however, that subscripted quantities in heavy type are vectors and not the scalar components of vectors.

We define the outer product of two vectors in n-space by the relation

$$(1)\quad \mathbf{A}_1 \times \mathbf{A}_2 = e\epsilon_{12j_3\cdots j_n}\mathbf{i}_{j_3}\cdots\mathbf{i}_{j_n}A_1A_2\sin(\mathbf{A}_1,\mathbf{A}_2),\quad (j_k = 3,\cdots,n),$$

where $\mathbf{i}_j$ are again mutually orthogonal unit vectors normal to $\mathbf{A}_1$ and $\mathbf{A}_2$, that is, vectors such that

$$(2)\qquad \mathbf{i}_{j_k}\cdot\mathbf{i}_{j_m} = \delta^k_m,\qquad \mathbf{i}_{j_k}\cdot\mathbf{A}_1 = 0,\qquad \mathbf{i}_{j_k}\cdot\mathbf{A}_2 = 0.$$

The symbols $\mathbf{i}_j$ will in the sequel always indicate such a set of mutually orthogonal unit vectors. From (1) it follows that

$$\mathbf{A}_1 \times \mathbf{A}_2 = -\mathbf{A}_2 \times \mathbf{A}_1,\qquad \mathbf{A}_i \times \mathbf{A}_i = 0,\quad (i \text{ not summed}),$$

and if $\mathbf{A}_1$ and $\mathbf{A}_2$ are the orthogonal vectors $\mathbf{i}_{j_1}$ and $\mathbf{i}_{j_2}$,

$$(3)\qquad \mathbf{i}_{j_1} \times \mathbf{i}_{j_2} = e\epsilon_{j_1\cdots j_n}\mathbf{i}_{j_1}\cdots\mathbf{i}_{j_n},\qquad (j_k = 1,\cdots,n).$$

3. **Polyadic products.** We now extend the notation of Gibbs[6] for double dot and double cross products. According to Gibbs

$$\mathbf{ab}:\mathbf{cd} = \mathbf{a}\cdot\mathbf{cb}\cdot\mathbf{d},\qquad \mathbf{ab}\overset{\times}{\times}\mathbf{cd} = \mathbf{a}\times\mathbf{cb}\times\mathbf{d}.$$

Likewise the double dot or double cross product of two dyadics $\Phi : \Psi$ or $\Phi\overset{\times}{\times}\Psi$, indicates the corresponding products arising from formally expanding these dyadics according to the distributive law. We extend the meaning of this notation to include the triple, quadruple, etc., products of any number of vectors, dyadics, or polyadics, the factors of the polyad of lower order, as they appear, to be applied in order to the polyad of higher order. Thus:

$$\mathbf{abcd}:\mathbf{efg} = \mathbf{ab}\cdot\mathbf{ec}\cdot\mathbf{fd}\cdot\mathbf{g},\qquad \mathbf{ab}:\mathbf{cdefg}\overset{\times}{\times}\mathbf{hk} = \mathbf{a}\cdot\mathbf{cb}\cdot\mathbf{def}\times\mathbf{hg}\times\mathbf{k},$$

and

$$\mathbf{abc}:\Gamma_\alpha\Gamma_\beta = \mathbf{ab}\cdot\Gamma_\alpha\mathbf{c}\cdot\Gamma_\beta,\qquad \mathbf{a}\cdot\Gamma_\alpha\mathbf{b}\overset{\times}{\times}\Gamma_\beta\mathbf{c} = \mathbf{a}\cdot\Gamma_\alpha\times\Gamma_\beta\mathbf{b}\times\mathbf{c}.$$

where Γ_i is a polyadic of order i.

From (3) we can now write with the aid of (2)

$$(4)\qquad \mathbf{i}_{k_1}\cdots\mathbf{i}_{k_p}:\mathbf{i}_{j_1}\times\mathbf{i}_{j_2}:\mathbf{i}_{l_1}\cdots\mathbf{i}_{l_q} = e\epsilon_{j_1j_2k_1\cdots k_pj_3\cdots j_sl_1\cdots l_q}\mathbf{i}_{j_3}\cdots\mathbf{i}_{j_s},$$
$$(r = p+3,\ s = n-q).$$

[6] Gibbs-Wilson, Vector Analysis, pp. 306–309.

The value of the right member evidently does not depend upon the position of the cross in the left member except for sign. Expressing any p vectors $\mathbf{A}_\beta$ in terms of the n vectors $\mathbf{i}_j$,

$$\mathbf{A}_\beta = \alpha_{\beta j}\mathbf{i}_j, \qquad (\beta = 1, \cdots, p, j = 1, \cdots, n),$$

we have with the aid of (4)

$$\mathbf{A}_1 \times \mathbf{A}_2 : \mathbf{A}_3 \cdots \mathbf{A}_p = \alpha_{1j_1}\alpha_{2j_2}\alpha_{3j_3} \cdots \alpha_{pj_p}\, \mathbf{i}_{j_1} \times \mathbf{i}_{j_2} : \mathbf{i}_{j_3} \cdots \mathbf{i}_{j_p}, \qquad (p \leqq n),$$

(5)

$$= e\epsilon_{j_1 \cdots j_n}\, \mathbf{i}_{j_1} \cdots \mathbf{i}_{j_s}\, \alpha_{1j_{s+1}} \cdots \alpha_{pj_n}, \qquad (s = n - p).$$

The latter result can be expressed in the determinant form

$$\mathbf{A}_1 \times \mathbf{A}_2 : \mathbf{A}_3 \cdots \mathbf{A}_p = e \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \cdots & \mathbf{i}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{i}_1 & \mathbf{i}_2 & \cdots & \mathbf{i}_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pn} \end{vmatrix}. \tag{6}$$

It follows immediately from consideration of the subscripts of (4) and (5):

In any polyadic product (including polyadics of order one, i.e., *vectors) formed by the double dot multiplication of the cross product of any two vectors with m, $(m = n - 2)$, other vectors, arranged in any way as prefactors and postfactors, the value of the polyadic without regard to sign is independent of the position of the cross. If the number of vector factors s in each term of the resulting polyadic,* i.e., *$s = n - 2 - m$, is even, changing the position of the cross does not change the sign; if s is odd the sign is changed or not changed depending upon whether the number of vectors crossed over is odd or even.*

According to the definition of the cross product it is at once evident that if any vector $\mathbf{A}_\beta$ in the product (5) is repeated, or if any of these vectors are linearly related, the product will vanish. If the polyadic product is of order one the result is a vector orthogonal to each of the vectors $\mathbf{A}_\beta$. It is accordingly called the vector product. If the polyadic product is of order zero, that is, if $n - p = 0$, the product is a scalar which we will call the *n-fold scalar product.*

4. **The n-fold scalar product.** If the vectors $\mathbf{A}_\beta$ are expressed in terms of any set of n linearly independent vectors $\mathbf{b}_i$,

$$\mathbf{A}_\beta = \alpha_{\beta i}\mathbf{b}_i,$$

we have for the n-fold scalar product

$$\mathbf{A}_1 \times \mathbf{A}_2 : \mathbf{A}_3 \cdots \mathbf{A}_n = |\, \alpha_{\beta j} \,|\, \mathbf{b}_1 \times \mathbf{b}_2 : \mathbf{b}_3 \cdots \mathbf{b}_n. \tag{7}$$

Since the cross may be placed in any position in this product without altering its value, we will write it, in analogy with the triple scalar product of 3-space, in the form

$$\mathbf{A}_1 \times \mathbf{A}_2 : \mathbf{A}_3 \cdots \mathbf{A}_n = [\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n].$$

The cross in the n-fold scalar product may be placed between any two vectors without altering the scalar's value; if the factors are interchanged the sign of the product alone may be changed, the sign of the new product being determined by the number of inversions necessary to restore the product to its original form. If the vectors $\mathbf{b}_i$ are the unit vectors $\mathbf{i}_i$ then

$$[\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n] = e\,|\,\alpha_{\beta j}\,|. \tag{8}$$

We will now select the value of e so that

$$[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_n\mathbf{A}_{n-1} \cdots \mathbf{A}_3] = e[\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n],$$

which defines e as

$$e = (-1)^{\frac{(n-2)(n-3)}{2}}.$$

5. **Reciprocal vectors.** Let any vector $\mathbf{B}$ be expressed in terms of the n linearly independent vectors $\mathbf{b}_i$,

$$\mathbf{B} = \beta_i\mathbf{b}_i.$$

To determine the coefficients β_i multiply both members by

$$\cdot\, \mathbf{b}_{j+1} \times \mathbf{b}_{j+2} : \mathbf{b}_{j+3} \cdots \mathbf{b}_n\mathbf{b}_1 \cdots \mathbf{b}_{j-1}.$$

All the terms of the right member will vanish except the one for which $i = j$; hence, solving for β_j,

$$\beta_j = \frac{[\mathbf{B}\mathbf{b}_{j+1} \cdots \mathbf{b}_n\,\mathbf{b}_1 \cdots \mathbf{b}_{j-1}]}{[\mathbf{b}_j\mathbf{b}_{j+1} \cdots \mathbf{b}_n\,\mathbf{b}_1 \cdots \mathbf{b}_{j-1}]}.$$

Substituting this value back in the above equation, we obtain

$$\mathbf{B} = \sum_j \frac{[\mathbf{B}\mathbf{b}_{j+1} \cdots \mathbf{b}_n \mathbf{b}_1 \cdots \mathbf{b}_{j-1}]}{[\mathbf{b}_j \cdots \mathbf{b}_n \mathbf{b}_1 \mathbf{b}_{j-1}]} \mathbf{b}_j$$

(10)

$$= \sum_j \frac{[\mathbf{b}_1 \cdots \mathbf{b}_{j-1} \mathbf{B}\mathbf{b}_{j+1} \cdots \mathbf{b}_n]}{[\mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_n]} \mathbf{b}_j.$$

The set of n vectors

$$(11) \qquad \frac{\mathbf{b}_{j+1} \times \mathbf{b}_{j+2} : \mathbf{b}_{j+3} \cdots \mathbf{b}_n \mathbf{b}_1 \cdots \mathbf{b}_{j-1}}{[\mathbf{b}_j \cdots \mathbf{b}_n \mathbf{b}_1 \cdots \mathbf{b}_{j-1}]} = \mathbf{b}_j', \qquad (n > 2),$$

will be called the vector reciprocals of the set $\mathbf{b}_j$ and will be denoted by $\mathbf{b}_j'$. Obviously the vectors $\mathbf{b}_j$ are also the reciprocals of the set $\mathbf{b}_j'$. We may now write equation (10) in either of two ways

$$(12) \qquad \mathbf{B} = \mathbf{B} \cdot \mathbf{b}_j \mathbf{b}_j', \qquad \mathbf{B} = \mathbf{B} \cdot \mathbf{b}_j' \mathbf{b}_j.$$

The dyadic $\mathbf{b}_j \mathbf{b}_j'$ or $\mathbf{b}_j' \mathbf{b}_j$ is therefore the idemfactor of the space and will be denoted by $\mathbf{I}$. A superscript to the left will be added, thus, ${}^n\mathbf{I}$, when it is desirable to indicate the dimensions of the space for which it is the idemfactor. From (11) it is evident that $\mathbf{i}_j' = \mathbf{i}_j$, hence $\mathbf{I}$ can be written

$$(13) \qquad \mathbf{I} = \mathbf{b}_j \mathbf{b}_j' = \mathbf{b}_j' \mathbf{b}_j = \mathbf{i}_j \mathbf{i}_j.$$

The last expression for $\mathbf{I}$ is easily shown to hold for $n = 2$, and we can now define the reciprocal vectors to $\mathbf{b}_1$ and $\mathbf{b}_2$ for 2-space.

$$(14) \qquad \mathbf{b}_1' = \frac{\mathbf{I} \times \mathbf{b}_2}{\mathbf{b}_1 \times \mathbf{b}_2}, \qquad \mathbf{b}_2' = \frac{\mathbf{I} \times \mathbf{b}_1}{\mathbf{b}_2 \times \mathbf{b}_1}.$$

It is to be recalled that the cross product is in this case a scalar. Equations (12) and (13) thus hold for $n = 2$. It follows at once from the definition of reciprocal vectors that

$$\mathbf{b}_i \cdot \mathbf{b}_j = \delta_j^i.$$

Let $\mathbf{A}_\alpha (\alpha = 1, \cdots, m)$ be a set of m linearly independent vectors lying in a space V_m immersed in a space V_n. Then any reciprocal vector of this set, say $\mathbf{A}_1'$, is given by

$${}^m\mathbf{A}_1' = \frac{{}^m(\mathbf{A}_2 \times \mathbf{A}_3) : \mathbf{A}_4 \cdots \mathbf{A}_m}{[\mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_m]},$$

where the superscript to the left of an expression again indicates the space for which the relation holds. Designate by

$$\mathbf{B}_\beta (\beta = m + 1, \cdots, n)$$

a set of vectors in V_n such that

$$\mathbf{B}_\beta \cdot \mathbf{A}_\alpha = 0, \qquad {}^n(\mathbf{B}_{m+1} \times \mathbf{B}_{m+2}) : \mathbf{B}_{m+3} \cdots \mathbf{B}_n \neq 0$$

then the n-fold scalar product,

$$[\mathbf{A}_1 \cdots \mathbf{A}_m \mathbf{B}_{m+1} \cdots \mathbf{B}_n],$$

does not vanish and there exists a reciprocal vector

$${}^n\mathbf{A}_1' = \frac{{}^n(\mathbf{A}_2 \times \mathbf{A}_3) : \mathbf{A}_4 \cdots \mathbf{A}_m \mathbf{B}_{m+1} \cdots \mathbf{B}_n}{[\mathbf{A}_1 \cdots \mathbf{A}_m \mathbf{B}_{m+1} \cdots \mathbf{B}_n]}.$$

From the definition of the cross product the numerator of the right member must be a vector $\mathbf{u}$ which lies in V_m and is orthogonal to the vectors $\mathbf{A}_\alpha (\alpha \neq 1)$. The denominator is $\mathbf{A}_1 \cdot \mathbf{u}$. Likewise the numerator of the expression for ${}^m\mathbf{A}'$ is a vector $\mathbf{v}$ in V_m orthogonal to the vectors $\mathbf{A}_\alpha (\alpha \neq 1)$ and is therefore parallel to $\mathbf{u}$. The denominator is $\mathbf{A}_1 \cdot \mathbf{v}$. It follows at once that

$$(15) \qquad {}^n\mathbf{A}_\alpha' = {}^m\mathbf{A}_\alpha'.$$

Since the vectors $\mathbf{i}_j$ of (5) can be expressed in terms of any set of $n - p$ vectors orthogonal to $\mathbf{A}_\beta (\beta = 1, \cdots, p)$, it follows that this polyadic product can be expressed in the form

$$\mathbf{A}_{s+1} \times \mathbf{A}_{s+2} : \mathbf{A}_{s+3} \cdots \mathbf{A}_n = e e_{\alpha_1 \cdots \alpha_s} \mathbf{u}_{\alpha_1} \cdots \mathbf{u}_{\alpha_s}, \ (\alpha = 1, \cdots, s = n - p),$$

where $\mathbf{A}_\beta \cdot \mathbf{u}_\alpha = 0$ and the $\mathbf{u}$'s are not necessarily orthogonal to each other. It therefore follows, since

$$\mathbf{A}_1 \cdots \mathbf{A}_s : \mathbf{A}_{s+1} \times \mathbf{A}_{s+2} : \mathbf{A}_{s+3} \cdots \mathbf{A}_n = [\mathbf{A}_1 \cdots \mathbf{A}_n],$$

that

$$(16) \quad A_{s+1} \times \mathbf{A}_{s+2} : \mathbf{A}_{s+3} \cdots \mathbf{A}_n = e_{\alpha_1 \cdots \alpha_s} \mathbf{A}_{\alpha_1}' \cdots \mathbf{A}_{\alpha_s}' [\mathbf{A}_1 \cdots \mathbf{A}_n], \qquad (\alpha = 1, \cdots, s).$$

Likewise

$$(17) \quad \mathbf{A}_{s+1}' \times \mathbf{A}_{s+2}' : \mathbf{A}_{s+3}' \cdots \mathbf{A}_n' = e_{\alpha_1 \cdots \alpha_s} \mathbf{A}_{\alpha_1} \cdots \mathbf{A}_{\alpha_s} [\mathbf{A}_1' \cdots \mathbf{A}_n'].$$

If the product $[\mathbf{A}_1' \cdots \mathbf{A}_n']$ be expanded by the use of (11) and (5), we obtain after simplification

$$(18) \qquad [\mathbf{A}_1' \cdots \mathbf{A}_n'][\mathbf{A}_1 \cdots \mathbf{A}_n] = 1.$$

By means of this relation it may be shown[7] that

$$[\mathbf{A}_1 \cdots \mathbf{A}_n][\mathbf{b}_1 \cdots \mathbf{b}_n] = |\mathbf{A}_i \cdot \mathbf{b}_j|, \qquad (i, j = 1, \cdots, n). \tag{19}$$

6. **The outer product redefined.** For 2-space

$$\mathbf{l} \times \mathbf{m} = lm \sin(\mathbf{l}, \mathbf{m}) = [\mathbf{lm}],$$

and we could have defined the outer product for 4-space by the relations

$$\mathbf{A} \times \mathbf{B} = \mathbf{lm} - \mathbf{ml}, \qquad [\mathbf{lm}] = [\mathbf{AB}].$$

For 5-space we would then have

$$\mathbf{A} \times \mathbf{B} = \epsilon_{ijk}\mathbf{l}_i\mathbf{l}_j\mathbf{l}_k, \qquad [\mathbf{l}_1\mathbf{l}_2\mathbf{l}_3] = [\mathbf{AB}],$$

and thus by the inductive process we can define the outer product for n-space by the equations

$$\mathbf{A}_1 \times \mathbf{A}_2 = \epsilon_{12i_3\cdots i_n}\mathbf{l}_{i_3} \cdots \mathbf{l}_{i_n}, \qquad {}^{n-2}[\mathbf{l}_3 \cdots \mathbf{l}_n] = [\mathbf{A}_1\mathbf{A}_2]. \tag{20}$$

This definition agrees with that given by (1) for, placing $\mathbf{l}_i = \alpha_{ij}\mathbf{i}_j$,

$$\mathbf{A}_1 \times \mathbf{A}_2 = \epsilon_{12i_3\cdots i_n}\alpha_{i_3j_3} \cdots \alpha_{i_nj_n}\mathbf{i}_{j_3} \cdots \mathbf{i}_{j_n} = |\alpha_{ij}|\epsilon_{12j_3\cdots j_n}\mathbf{i}_{j_3} \cdots \mathbf{i}_{j_n},$$

and with the aid of (8)

$$|\alpha_{ij}| = e[\mathbf{l}_3 \cdots \mathbf{l}_n] = e[\mathbf{A}_1\mathbf{A}_2] = eA_1A_2 \sin(\mathbf{A}_1, \mathbf{A}_2).$$

7. **Functions of a dyadic, the invariants.** We now define $n - 1$ functions of a dyadic Φ, which are extensions of Gibbs' *second* and *third*[8] of a dyadic. Let

$$\Phi = \mathbf{b}_i\mathbf{c}_i, \qquad (i = 1, \cdots, m \leqq n),$$

then the *pth* of Φ is defined by

$$\Phi_p = (1/p!)\Phi \overset{\times}{\times} \Phi : \Phi^{p-2}, \tag{21}$$

where Φ^{p-2} indicates $\Phi\Phi \cdots$ to a total of $p - 2$ factors. When the right member is expanded, this becomes

$$\Phi_p = (1/p!)(\mathbf{b}_{i_1} \times \mathbf{b}_{i_2} : \mathbf{b}_{i_3} \cdots \mathbf{b}_{i_p}\mathbf{c}_{i_1} \times \mathbf{c}_{i_2} : \mathbf{c}_{i_3} \cdots \mathbf{c}_{i_p}), \tag{22}$$

$$(i_k = 1, \cdots, m).$$

In particular for the *nth* of Φ we have the scalar

$$\Phi_n = (1/n!)\Phi \overset{\times}{\times} \Phi : \Phi^{n-2} = [\mathbf{b}_1 \cdots \mathbf{b}_n][\mathbf{c}_1 \cdots \mathbf{c}_n],$$

[7] Gibbs-Wilson, p. 56.

[8] Gibbs-Wilson, pp. 310–311.

and if the consequents of Φ are selected as the vectors $\mathbf{i}_k$ we have from (8)

$$\Phi_n = e[\mathbf{b}_1 \cdots \mathbf{b}_n] = |\beta_{ij}|, \qquad \mathbf{b}_i = \beta_{ij}\mathbf{i}_j, \tag{23}$$

which identifies Φ_n as the determinant of the dyadic. If we define the *rank* of a dyadic as numerically equal to the least number of dyads in which it may be expressed, it follows at once from (22):

THEOREM. *The necessary and sufficient condition that a dyadic Φ be of rank m is that $\Phi_m \neq 0$, $\Phi_{m+1} = 0$.*

If we select $\mathbf{c}_j = \mathbf{i}_j$ and $\mathbf{b}_i = \beta_{ij}\mathbf{i}_j$ then (22) becomes with the aid of (6) and (4)

$$\Phi_p = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \cdots & \mathbf{i}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{i}_1 & \mathbf{i}_2 & \cdots & \mathbf{i}_n \\ \beta_{1i_1} & \beta_{1i_2} & \cdots & \beta_{1i_n} \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{pi_1} & \beta_{pi_2} & \cdots & \beta_{pi_n} \end{vmatrix} \epsilon_{i_1 \cdots i_s j_1 \cdots j_p} \mathbf{i}_{i_1} \cdots \mathbf{i}_{i_s}, \qquad (s = n - p), \quad (i, j = 1, \cdots, n).$$

which can vanish only if all the p-rowed determinants in the β's vanish. Hence it follows at once with the aid of (23) that *the rank of a dyadic is identical with the rank of its determinant.*

We have for the pth of a linear combination of two dyadics

$$p!(\lambda\Phi + \mu\Psi)_p = (\lambda\Phi + \mu\Psi) \overset{\times}{\times} (\lambda\Phi + \mu\Psi) : (\lambda\Phi + \mu\Psi)^{p-2}.$$

Since the order of the dyadic factors and the double dot and double cross may obviously be interchanged in any manner, the right member may be expanded into the form

$$\Phi \overset{\times}{\times} \Phi : \Phi^{p-2}\lambda^p + \sum_{r=1}^{p-2} \binom{p}{r} \Phi \overset{\times}{\times} \Phi : \Phi^{p-r-2}\Psi^r\lambda^{p-r}\mu^r + \cdots + \binom{p}{p-1} \Phi : \Psi \overset{\times}{\times} \Psi^{p-2}\lambda\mu^{p-1} + \Psi \overset{\times}{\times} \Psi : \Psi^{p-2}\mu^p,$$

whence

$$(\lambda\Phi + \mu\Psi)_p = \Phi_p\lambda^p + \sum_{r=1}^{p-2} \frac{1}{r!} \Phi_{p-r} : \Psi^r\lambda^{p-r}\mu^r + \Phi : \Psi_{p-1}\lambda\mu^{p-1} + \Psi_p\mu^p.$$

For $\lambda = 1$, $\mu = -\lambda$, $\Psi = \mathbf{I}$, $p = n$, we have the so called characteristic dyadic

$$(\Phi - \lambda\mathbf{I})_n = \Phi_n + \sum_{r=1}^{n-2} \frac{(-1)^r}{r!} \Phi_{n-r} : \mathbf{I}^r\lambda^r + \frac{(-1)^{n-1}}{n-1} \mathbf{I}^{n-1}\lambda^{n-1} + (-1)^n\mathbf{I}_n\lambda^n,$$

since $\mathbf{I}_{n-1} = 1/(n-1)\mathbf{I}^{n-1}$. The coefficient $\mathbf{I}_n$ is equal to unity; the remaining coefficients of the right member constitute n scalar invariants of the dyadic Φ and will, with the exception of Φ_n, be designated thus, $(\Phi_p)_s$, where we have adopted the notation of Gibbs,[9] since they are obvious generalizations of the three principal invariants pointed out by him. Let $n - r = p$; then by the aid of (13) and (22) for $m = n$ these invariants expressed in terms of the vectors $\mathbf{b}_i$ and $\mathbf{c}_i$ are

$$\Phi_n = [\mathbf{b}_1 \cdots \mathbf{b}_n][\mathbf{c}_1 \cdots \mathbf{c}_n], \tag{24}$$

$$(\Phi_p)_s = \frac{1}{r!}\Phi_p{:}\mathbf{I}^r = \frac{1}{r!\,p!}[\mathbf{b}_{i_1} \cdots \mathbf{b}_{i_p}\mathbf{b}_{j_1} \cdots \mathbf{b}_{j_r}][\mathbf{c}_{i_1} \cdots \mathbf{c}_{i_p}\mathbf{b}'_{j_1} \cdots \mathbf{b}'_{j_r}], \tag{25}$$

$$(i, j = 1, \cdots, n).$$

When (16) is applied to the latter equation it can be written

$$(\Phi_p)_s = \frac{1}{p!}(\mathbf{c}_{i_1} \times \mathbf{c}_{i_2}{:}\mathbf{c}_{i_3} \cdots \mathbf{c}_{i_p}){:}(\mathbf{b}_{i_1} \times \mathbf{b}_{i_2}{:}\mathbf{b}_{i_3} \cdots \mathbf{b}_{i_p}), \tag{25$'$}$$

or if we apply (17) and (18) instead of (16), we obtain

$$(\Phi_p)_s = \frac{1}{r!\,p!}\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p}\mathbf{c}_{i_1} \cdots \mathbf{c}_{i_p}{:}\mathbf{b}_{j_1} \cdots \mathbf{b}_{j_p}, \tag{25$''$}$$

$$(p = 1, \cdots, n - r;\ i, j = 1, \cdots, n).$$

The characteristic equation may now be written in the form

$$(\Phi - \lambda\mathbf{I})_n = (-1)^n\lambda^n + \sum_{p=1}^{n-1}(-1)^r(\Phi_p)_s\lambda^{n-p} + \Phi_n.$$

This relation can be shown to be satisfied when λ is replaced by Φ giving the Hamilton-Cayley equation for n-space.

$$\Phi^n + \sum_{p=1}^{n-1}(-1)^p(\Phi_p)_s\Phi^{n-p} + \Phi_n\mathbf{I} = 0, \tag{26}$$

where now $\Phi^n = \Phi{\cdot}\Phi{\cdot} \cdots$ to a total of n factors.

8. **The derivatives, divergence, curl.** We define the gradient of a scalar function ϕ by the relation

$$d\phi = d\mathbf{r}{\cdot}\nabla\phi$$

[9] Gibbs-Wilson, pp. 319–320.

from which it follows that $\nabla\phi$ is normal to any direction of $d\mathbf{r}$ for which ϕ is constant; that is, it is normal to the hypersurface $\phi(\mathbf{r}) = c$ through the current point. The unit normal $\mathbf{n}$ in the direction of $\nabla\phi$ is given by

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 \cdots \mathbf{a}_{n-1}}{[\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_{n-1}]},$$

where $\mathbf{a}_i$ are any $n - 1$ independent vectors tangent to this hypersurface. The dyadic derivative $\nabla\mathbf{F}$ of a vector function $\mathbf{F}$ of a vector $\mathbf{r}$ is defined by the relation

$$d\mathbf{F} = d\mathbf{r} \cdot \nabla\mathbf{F}. \tag{27}$$

The derivatives $\nabla\phi$ and $\nabla\mathbf{F}$ thus defined hold for any space and are independent of any coordinate system. They can, however, be easily expressed in terms of such systems. Since any dyadic of rank n may be expressed as the sum of n dyads of which either the antecedents or the consequents may be arbitrarily chosen,[10] we select as the consequents the vector functions $\partial\mathbf{F}/\partial u_i$ $(i = 1, \cdots, n)$, where u_i are the curvilinear coordinates of the space. $\nabla\mathbf{F}$ and its conjugate $\mathbf{F}\nabla$ thus become

$$\nabla\mathbf{F} = \mathbf{a}_i \, \partial\mathbf{F}/\partial u_i, \qquad \mathbf{F}\nabla = \partial\mathbf{F}/\partial u_i \mathbf{a}_i, \tag{28}$$

where the vectors $\mathbf{a}_i$ are determined by the relations

$$d\mathbf{F} = d\mathbf{r} \cdot \nabla\mathbf{F}, \qquad d\mathbf{r} = \partial\mathbf{r}/\partial u_i du_i.$$

It is evident that the vectors $\mathbf{a}_i$ are the reciprocal vectors to the set $\partial\mathbf{r}/\partial u_i$. Likewise the gradient can be written

$$\nabla\phi = \mathbf{a}_i \partial\phi/\partial u_i. \tag{29}$$

The divergence $\nabla \cdot \mathbf{F}$ and the curl $\nabla \times \mathbf{F}$ of the vector function $\mathbf{F}(\mathbf{r})$ are defined as those functions of $\nabla\mathbf{F}$ which are obtained by taking the scalar and the cross, respectively, of $\nabla\mathbf{F}$. Thus

$$\nabla \cdot \mathbf{F} = (\nabla\mathbf{F})_. = \mathbf{a}_i \cdot \partial\mathbf{F}/\partial u_i, \tag{30}$$

$$\nabla \times \mathbf{F} = (\nabla\mathbf{F})_\times = \mathbf{a}_i \times \partial\mathbf{F}/\partial u_i. \tag{31}$$

From (27) we see that $d\mathbf{r} = d\mathbf{r} \cdot \nabla\mathbf{r}$ for any direction of $d\mathbf{r}$. Hence

$$\nabla\mathbf{r} = \mathbf{I}, \qquad \nabla \cdot \mathbf{r} = \mathbf{I}_. = n,$$

[10] Gibbs-Wilson, p. 271.

where $\mathbf{I}$ is the idemfactor of the space to which $\mathbf{r}$ is limited. If $\mathbf{r}$ is a vector $\mathbf{c}$ constant in magnitude but variable in direction then, since in the relation $d\mathbf{c} = d\mathbf{c}\cdot\nabla_{\mathbf{c}}\mathbf{c}$[11] $d\mathbf{c}$ is everywhere orthogonal to $\mathbf{c}$,

$$\nabla_{\mathbf{c}}\mathbf{c} = {}^{n-1}\mathbf{I} = {}^{n}\mathbf{I} - \mathbf{cc}/c^2, \qquad \nabla_{\mathbf{c}}\cdot\mathbf{c} = n - 1, \tag{32}$$

where ${}^{n-1}\mathbf{I}$ is the idemfactor of the $n - 1$ space over which $\mathbf{c}$ varies.

9. **Vector and dyadic derivatives in a subspace.** Let us consider the m-space V_m determined by the intersection of the p, $(p = n - m)$, hypersurfaces

$$\phi_\alpha(\mathbf{r}) = c_\alpha, \qquad (\alpha = 1, \cdots, p < n - 1),$$

immersed in an n-space V_n. Construct m independent scalar functions

$$\phi_\beta(\mathbf{r}) = u_\beta, \qquad (\beta = p + 1, \cdots, n),$$

which are uniform and continuous and possess continuous gradients and for which the scalar product

$$\Delta(\mathbf{r}) = [\nabla\phi_1 \cdots \nabla\phi_p\nabla\phi_{p+1} \cdots \nabla\phi_n] \neq 0 \tag{33}$$

in the neighborhood of the current point $\mathbf{r}$. The n equations $\phi_\alpha = c_\alpha$, $\phi_\beta = u_\beta$, according to the writer's theorem[12] on implicit functions of vector variables, may now be solved for $\mathbf{r}$ in this neighborhood.

$$\mathbf{r} = \mathbf{r}(c_1, \cdots, c_p, u_{p+1}, \cdots, u_n). \tag{34}$$

If the c's as well as the u's are considered as independent variables, this equation defines the space V_n; if the c's are regarded as constants then (34) defines the space V_m, in which case the vector $\mathbf{r}$ will be written with a bar, thus

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}(u_{p+1}, \cdots, u_n). \tag{35}$$

Evidently

$$\nabla_{\mathbf{r}}\mathbf{r} = \mathbf{I}, \qquad \nabla_{\bar{\mathbf{r}}}\bar{\mathbf{r}} = \nabla_{\bar{\mathbf{r}}}\mathbf{r} = {}^{m}\mathbf{I}, \tag{36}$$

[11] The notation $\nabla_{\mathbf{r}}\phi$, $\nabla_{\mathbf{c}}\mathbf{F}$, etc., will be used when it is desired to call attention to the particular independent variable with respect to which the derivative is taken, as is done in ordinary scalar differentiation. The notation $f_{\mathbf{r}}$, $f_{\mathbf{rs}}$, etc., will be used to denote partial differentiation.

[12] Existence Theorem on implicit vector functions, Jour. Math. & Physics, Vol. XV, p. 97 (1936).

where $\mathbf{I}$ is the idemfactor of V_n and $^m\mathbf{I}$ the idemfactor of the V_m. Differentiating $\mathbf{r}$ and $\bar{\mathbf{r}}$ with respect to s, we have

$$\mathbf{t} = c_\alpha' \mathbf{r}_\alpha + u_\beta' \mathbf{r}_\beta, \qquad \bar{\mathbf{t}} = u_\beta' \bar{\mathbf{r}}_\beta,$$

where $\mathbf{r}_\alpha = \partial \mathbf{r}/\partial c_\alpha$, $\mathbf{r}_\beta = \partial \mathbf{r}/\partial u_\beta$, and $\mathbf{t}$ is the unit tangent to any curve in V_n and $\bar{\mathbf{t}}$ the unit tangent to any curve in V_m. With $\mathbf{t}$ and $\bar{\mathbf{t}}$ thus defined we have from (32)

$$\nabla_t \mathbf{t} = \mathbf{I} - \mathbf{tt}, \qquad \nabla_{\bar{t}} \bar{\mathbf{t}} = {}^m\mathbf{I} - \bar{\mathbf{t}}\bar{\mathbf{t}}.$$

Since both antecedents and consequents of $^m\mathbf{I}$ lie[13] in V_m it is evident *that* $\nabla_{\bar{r}} \mathbf{r}$ *and* $\nabla_{\bar{t}} \bar{\mathbf{t}}$ *when applied as prefactor or postfactor to any vector act as annihilators to any component of such vector which is orthogonal to* V_m. *Let* $\mathbf{P}$ *be any vector in* V_n; *then* $\nabla_{\bar{r}} \mathbf{r} \cdot \mathbf{P} = \mathbf{P} \cdot \nabla_{\bar{r}} \mathbf{r}$ *is the projection of* $\mathbf{P}$ *on* V_m. $\nabla_{\bar{t}} \bar{\mathbf{t}}$ *in addition annihilates all components parallel to* $\mathbf{t}$. $\nabla_{\bar{t}} \bar{\mathbf{t}} \cdot \mathbf{P} = \mathbf{P} \cdot \nabla_{\bar{t}} \bar{\mathbf{t}}$ *is thus the projection of* $\mathbf{P}$ *upon the* $m - 1$ *space orthogonal to* $\mathbf{t}$ *and lying in* V_m.

We wish to express these operators in terms of the functions ϕ_i, $(i = 1, \cdots, n)$, or their derivatives. Carrying out the differentiation of the equations (34) and (35) with respect to $\mathbf{r}$ and $\bar{\mathbf{r}}$ respectively, we have

$$(38) \quad \nabla_r \mathbf{r} = \mathbf{I} = \nabla_r \phi_i \mathbf{r}_i = \mathbf{r}_i \nabla_r \phi_i, \qquad \nabla_{\bar{r}} \mathbf{r} = {}^m\mathbf{I} = \nabla_{\bar{r}} \phi_\beta \mathbf{r}_\beta = \mathbf{r}_\beta \nabla_{\bar{r}} \phi_\beta,$$

from which it is evident that the vectors $\mathbf{r}_i$ are the reciprocals of the set $\nabla \phi_i$ in V_n and the vectors $\mathbf{r}_\beta$ are the reciprocals of the set $\nabla_{\bar{r}} \phi_\beta$ in V_m, and consequently

$$(39) \qquad \nabla_r \phi_i \cdot \mathbf{r}_j = \delta_j^i, \qquad \nabla_{\bar{r}} \phi_\beta \cdot \mathbf{r}_\gamma = \delta_\beta^\gamma, \qquad (i, j = 1, \cdots, n; \beta, \gamma = p + 1, \cdots n).$$

The vectors $\mathbf{r}_\beta$ thus lie in V_m orthogonal to $\nabla \phi_\alpha$. The system of vectors

$$\nabla \phi_1, \cdots, \nabla \phi_p, \mathbf{r}_{p+1}, \cdots, \mathbf{r}_n$$

owing to (18) and (33) therefore possesses a reciprocal system $\nabla' \phi_\alpha$, $\mathbf{r}_\beta'$, and we can write

$$\mathbf{I} = \nabla \phi_\alpha \nabla' \phi_\alpha + \mathbf{r}_\beta \mathbf{r}_\beta',$$

[13] The expression "lie in" or "belong to" as used herein is to be understood to include those vectors which are commonly spoken of as being tangent to the space.

whence, since $\mathbf{r}'_\beta$ as shown in Sec. 5 are also reciprocals to the set $\mathbf{r}_\beta$ in V_m, we have with the aid of (38)

$$\nabla_{\bar{\mathbf{r}}}\mathbf{r} = {}^{m}\mathbf{I} = \mathbf{r}_\beta\mathbf{r}'_\beta = \mathbf{I} - \nabla\phi_\alpha\nabla'\phi_\alpha, \tag{40}$$

$$\nabla_{\bar{\mathbf{t}}}\mathbf{t} = {}^{m}\mathbf{I} - \mathbf{tt} = \mathbf{I} - \nabla\phi_\alpha\nabla'\phi_\alpha - \mathbf{tt}. \tag{41}$$

If we multiply $(38)_2$ by $\cdot\nabla_r\phi_\beta$ and apply $(39)_1$ we obtain

$$\nabla_{\bar{\mathbf{r}}}\mathbf{r}\cdot\nabla_r\phi_\beta = \nabla_{\bar{\mathbf{r}}}\phi_\beta.$$

With the aid of this we have for any vector or scalar function F of $\mathbf{r}$ where $\mathbf{r}$ is defined by (35)

$$\nabla_{\bar{\mathbf{r}}}F = \nabla_{\bar{\mathbf{r}}}\phi_\beta\partial F/\partial u_\beta = \nabla_{\bar{\mathbf{r}}}\mathbf{r}\cdot\nabla_r\phi_\beta\partial F/\partial u_\beta.$$

Now considering F as a function of $\mathbf{r}$ defined by (34), wherein the c's are regarded as variables, and adding $\nabla_{\bar{\mathbf{r}}}\mathbf{r}\cdot\nabla_r\phi_\alpha\partial F/\partial c_\alpha = 0$ to the right member we have

$$\nabla_{\bar{\mathbf{r}}}F = \nabla_{\bar{\mathbf{r}}}\mathbf{r}\cdot(\nabla_r\phi_\alpha\partial F/\partial c_\alpha + \nabla_r\phi_\beta\partial F/\partial u_\beta) = \nabla_{\bar{\mathbf{r}}}\mathbf{r}\cdot\nabla_r F. \tag{42}$$

Similarly we obtain, for F a scalar or vector function of $\mathbf{t}$,

$$\nabla_{\bar{\mathbf{t}}}F = \nabla_{\bar{\mathbf{t}}}\mathbf{t}\cdot\nabla_t F \tag{43}$$

and for F a function of both $\mathbf{r}$ and $\mathbf{t}$

$$F_{\bar{\mathbf{r}}} = \nabla_{\bar{\mathbf{r}}}\mathbf{r}\cdot F_r, \qquad F_{\bar{\mathbf{t}}} = \nabla_{\bar{\mathbf{t}}}\mathbf{t}\cdot F_t. \tag{44}$$

The results (42), (43), and (44) are exactly those we would obtain if we proceeded to differentiate F implicitly regarding $\mathbf{r}$ as a function of $\bar{\mathbf{r}}$ and $\mathbf{t}$ as a function of $\bar{\mathbf{t}}$.

$$F[\mathbf{r}(\bar{\mathbf{r}})], \qquad F[\mathbf{t}(\bar{\mathbf{t}})], \qquad F[\mathbf{r}(\bar{\mathbf{r}}), \mathbf{t}(\bar{\mathbf{t}})].$$

If we let $c_\alpha = u_\alpha$ in (34) we can immediately express the derivative $\nabla_r F$ in terms of the curvilinear coordinates u_i of V_n by substituting the value of $\nabla\phi_i$ in terms of the reciprocal set $\mathbf{r}_i$. Thus, since

$$\nabla_r F = \nabla\phi_i\partial F/\partial u_i, \qquad (i = 1, \cdots, n),$$

it follows at once that

$$\nabla_r F = \sum_{i=1}^{n} \frac{\mathbf{r}_{i+1} \times \mathbf{r}_{i+2} \vdots \mathbf{r}_{i+3} \cdots \mathbf{r}_n\, \mathbf{r}_1 \cdots \mathbf{r}_{i-1}}{[\mathbf{r}_i \cdots \mathbf{r}_n\, \mathbf{r}_1 \cdots \mathbf{r}_{i-1}]} \frac{\partial F}{\partial u_i}. \tag{45}$$

This is a generalization of the expression obtained by Weatherburn[14] for curvilinear coordinates for 3-space. By applying (42) and (40) to (45) ∇F can be obtained in terms of the coordinates u_β of V_m.

[14] C. E. Weatherburn, On triple systems of surfaces and non-orthogonal curvilinear coordinates, Proc. Royal Soc. Edin., Vol. 46, (1926), pp. 194–205.

THE STABILITY OF QUADRATIC VELOCITY-DISTRIBUTIONS FOR AN INVISCID LIQUID FLOWING BETWEEN PARALLEL PLANES

By J. L. Synge[1]

This paper contains a proof that *the steady flow of an inviscid liquid with a velocity-distribution $u_0(y)$ between fixed planes $y = \pm b$ is stable if $u_0(y)$ is a quadratic function of y.* It is true that this result is given by Rayleigh's condition for stability,[2] viz., that d^2u_0/dy^2 shall be of one sign in the liquid, but the method by which that condition was derived is of such doubtful validity[3] that the present proof, which is very simple and direct, may be of interest. The present method is based on a quantity called the "measure of turbulence" of a disturbed motion, related to (and, in the case of the small disturbance of a quadratic velocity distribution, differing only by a constant factor from) a geometrical quantity called the "deviation" of a disturbed motion. The paper concludes with a criticism of the characteristic value method so widely employed in the theory of hydrodynamical stability: it is shown that the method leads to a contradiction under certain circumstances.

Let us consider the disturbance of a steady flow of an inviscid liquid between parallel planes $y = \pm b$. The disturbance is assumed to be either periodic in x or localised in the sense that, if ω' represents the vorticity of the disturbance,[4] then the integral

$$\int\int \omega'^2 \, dxdy,$$

taken over the whole region between the planes, converges. The disturbance may be infinitesimal or finite.

Let us write for the velocity-components and the vorticity of the disturbed motion

$$\begin{aligned} u(x, y, t) &= u_0(y) + u'(x, y, t), \\ v(x, y, t) &= v'(x, y, t), \qquad \omega(x, y, t) = \omega_0(y) + \omega'(x, y, t), \end{aligned} \tag{1}$$

[1] Professor of applied mathematics, University of Toronto.—Ed.

[2] Lord Rayleigh, Proc. London Math. Soc., 11 (1880), pp. 57–70. (Sci. Papers, 1, 474–487.)

[3] For a criticism of the method, see J. L. Synge, Phil. Trans. Roy. Soc. A 234 (1934), 43–78.

[4] That is, the excess (at the point x, y) of the vorticity of the disturbed motion over the vorticity of the steady motion.

where u_0, ω_0 refer to the undisturbed motion, and u', v', ω' to the superimposed disturbance. We have then

$$\omega_0 = -\frac{1}{2}\frac{du_0}{dy}, \tag{2}$$

$$\omega' = \frac{1}{2}\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right), \tag{3}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{4}$$

We have also the equation of conservation of vorticity,

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0, \tag{5}$$

or

$$\frac{\partial \omega'}{\partial t} + (u_0 + u')\frac{\partial \omega'}{\partial x} + v'\left(\frac{d\omega_0}{dy} + \frac{\partial \omega'}{\partial y}\right) = 0. \tag{6}$$

Let us define the *measure of turbulence* of the disturbed motion as

$$M(t) = \int\int_S \omega'^2\, dxdy, \tag{7}$$

where S is a fixed area. When the disturbance is periodic in x with period $2a$, we shall take S to be the rectangle bounded by $x = \pm a$, $y = \pm b$; when the disturbance is localised, we shall take S to be the whole region between the planes $y = \pm b$.

Let us consider first the case of a disturbance periodic in x. We have

$$\frac{dM}{dt} = 2\int\int_S \omega'\frac{\partial \omega'}{\partial t}\, dxdy, \tag{8}$$

or, by (6),

$$\frac{dM}{dt} = I_1 + I_2 + I_3 + I_4, \tag{9}$$

where

$$I_1 = -2\int\int_S \omega' u_0 \frac{\partial \omega'}{\partial x}\, dxdy,$$

$$
\begin{aligned}
I_2 &= -2\int\int_s \omega' u' \frac{\partial \omega'}{\partial x}\, dxdy, \\
I_3 &= -2\int\int_s \omega' v' \frac{d\omega_0}{dy}\, dxdy, \\
I_4 &= -2\int\int_s \omega' v' \frac{\partial \omega'}{\partial y}\, dxdy.
\end{aligned} \tag{10}
$$

Now, integrating with respect to x, we obtain

$$
I_1 = -\int_{-b}^{b} u_0 [\omega'^2]_{x=-a}^{x=a}\, dy = 0, \tag{11}
$$

on account of the assumed periodicity of the disturbance, and also

$$
\begin{aligned}
I_2 &= -\int_{-b}^{b} [u'\omega'^2]_{x=-a}^{x=a}\, dy + \int\int_s \omega'^2 \frac{\partial u'}{\partial x}\, dxdy \\
&= \int\int_s \omega'^2 \frac{\partial u'}{\partial x}\, dxdy,
\end{aligned} \tag{12}
$$

for the same reason. Substituting for ω' from (3) in the expression for I_3, and integrating by parts, we have

$$
\begin{aligned}
I_3 &= -\int\int_s \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) v' \frac{d\omega_0}{dy}\, dxdy \\
&= -\frac{1}{2}\int_{-b}^{b} \left[v'^2 \frac{d\omega_0}{dy}\right]_{x=-a}^{x=a} dy + \int_{-a}^{a} \left[u'v' \frac{d\omega_0}{dy}\right]_{y=-b}^{y=b} dx \\
&\quad -\int\int_s u' \frac{\partial v'}{\partial y} \frac{d\omega_0}{dy}\, dxdy - \int\int_s u'v' \frac{d^2\omega_0}{dy^2}\, dxdy.
\end{aligned} \tag{13}
$$

The first expression vanishes from the periodicity of the disturbance, and the second vanishes since $v' = 0$ for $y = \pm b$; the third integral also vanishes since, if we substitute $-\partial u'/\partial x$ for $\partial v'/\partial y$ from (4), we get

$$
\begin{aligned}
-\int\int_s u' \frac{\partial v'}{\partial y} \frac{d\omega_0}{dy}\, dxdy &= \int\int_s u' \frac{\partial u'}{\partial x} \frac{d\omega_0}{dy}\, dxdy \\
&= \int_b^b [\tfrac{1}{2} u'^2]_{x=-a}^{x=a} \frac{d\omega_0}{dy}\, dy = 0.
\end{aligned}
$$

Thus

$$
I_3 = -\int\int_s u'v' \frac{d^2\omega_0}{dy^2}\, dxdy. \tag{14}
$$

For I_4 we have

$$I_4 = -\int_{-a}^{a} [v'\omega'^n]_{y=-b}^{y=b}\, dx + \int\int_s \omega'^n \frac{\partial v'}{\partial y}\, dxdy$$
$$= \int\int_s \omega'^n \frac{\partial v'}{\partial y}\, dxdy. \tag{15}$$

Adding our results, and noting that

$$I_2 + I_4 = 0,$$

by virtue of (4), we have

$$\frac{dM}{dt} = -\int\int_s u'v' \frac{d^2\omega_0}{dy^2}\, dxdy = \tfrac{1}{2}\int\int_s u'v' \frac{d^3u_0}{dy^3}\, dxdy. \tag{16}$$

Thus if u_0 is of the form

$$u_0 = Ay^2 + By + C, \tag{17}$$

M is a constant.

Except for a trivial velocity of translation parallel to the walls, the distribution of vorticity determines the distribution of velocity. If $M = 0$, then $\omega' = 0$, and hence the disturbance vanishes. In the case of a finite initial disturbance, ω' is finite initially and so are u' and v'. It is difficult to see how stability is to be precisely defined in the case of a finite disturbance: it seems that it would be as good a definition as any to say that the motion is stable if the mean value of ω'^n remains less than an assigned value. *In this sense, flow with the quadratic profile* (17) *is stable with respect to a finite disturbance.*

In the case of an infinitesimal disturbance, we can examine the situation more closely. If the disturbance was originally of the order of ϵ (that is, if u' and v' were originally of this order), then, if the velocity profile is as in (17), M is permanently of the order of ϵ^2. Hence ω' is permanently of order ϵ, except perhaps in a small region in which it becomes finite: the area of this region will be of the order of ϵ^2. Now (omitting from consideration a trivial velocity of translation parallel to the walls) u' and v' are determined by ω'; the part of the region in which ω' is of order ϵ gives contributions of order ϵ to u' and v', whereas the small region only gives contributions of the order of ϵ^2. Hence *the quadratic profile* (17) *is stable with respect to small disturbances in the usual sense, u' and v' remaining small.*

If the disturbance is not periodic, but localised, we may follow exactly the same line of argument, putting $a = \infty$. Square brackets which

formerly vanished on account of periodicity, now vanish on account of the localisation of the disturbance.

The significance of the result is more clearly seen by considering a quantity which we may call the *deviation* of the motion, and which we shall now define. In the undisturbed motion the lines of constant vorticity are straight. Consider one of them (L) on which the vorticity has the value C, and consider the moving curve (L') in the disturbed motion for which the vorticity has this same value. The relation of the system of straight lines L to their correspondents L' gives us a graphical picture of the extent of the disturbance. Let y, $y + \eta$ be the ordinates on L, L' respectively corresponding to a certain abscissa x and time t. We define the *deviation* of the motion at time t to be

$$\delta(t) = \int\int_S [\eta(x, y, t)]^2 \, dxdy, \tag{18}$$

where the region S is defined as above for periodic and localised disturbances. If the disturbance is finite, it may well happen that there is no curve L' in the disturbed motion having the assigned vorticity C. Our definition is not valid for such a case, but it will be valid if the disturbed state is one which may be obtained from the steady motion with conservation of the vorticity point-by-point. In the case of an infinitesimal disturbance, this difficulty does not arise, since we may omit an infinitesimal layer next each wall from the region of integration of (18), an error of only the third order arising thereby.

Now

$$\omega_0(y + \eta) + \omega'(x, y + \eta, t) = \omega_0(y), \tag{19}$$

and hence, confining our attention to infinitesimal disturbances,

$$\eta = -\frac{\omega'(x, y, t)}{d\omega_0/dy}. \tag{20}$$

This is the Cartesian equation of L', the running coordinates being x, η. Although the measure of turbulence, defined in (7), is not in general equal to the deviation, defined in (18), they differ only by a constant factor when the velocity-distribution is of the form (17). *Hence when the velocity-distribution is a quadratic function of y, the deviation of the disturbed motion remains constant, the disturbance being infinitesimal.*

A remark may be added in criticism of the characteristic value method. The basis of the method is to assume a solution as a superposition of terms of the type

$$u' = f(y) \exp i\alpha(x - ct), \quad v' = g(y) \exp i\alpha(x - ct), \quad (c \neq 0), \tag{21}$$

each pair of expressions of this type separately satisfying the equations of motion and the boundary conditions. Now let us suppose that there is superimposed on a steady motion which has zero velocity on the walls a disturbance whose initial vorticity does *not* vanish on the walls, and let us assume for the subsequent motion a superposition of solutions of the type described above. The vorticity corresponding to (21) is

$$\omega' = \tfrac{1}{2}h(y) \exp i\alpha(x - ct), \qquad h(y) = i\alpha g(y) - df/dy. \tag{22}$$

When we substitute in the approximate form of (6), namely,

$$\frac{\partial \omega'}{\partial t} + u_0 \frac{\partial \omega'}{\partial x} + v' \frac{d\omega_0}{dy} = 0, \tag{23}$$

the above expressions for v', ω', and remove the exponential factor, we get

$$i\alpha(u_0 - c)h(y) - g(y)\, d^2u_0/dy^2 = 0. \tag{24}$$

Now, as stated above (and it is the very essence of the characteristic-value method) the v' of (21) is to satisfy the boundary conditions: $v' = 0$ for $y = \pm b$. Thus $g(y) = 0$ for $y = \pm b$. Also by hypothesis, $u_0 = 0$ for $y = \pm b$, and $c \neq 0$. Therefore $h(y) = 0$ for $y = \pm b$. But we deliberately choose a disturbance whose vorticity did *not* vanish on the walls. Hence we arrive at a contradiction and must conclude that, in this particular instance at any rate, the search for "typical solutions" such as (21) is useless, and conclusions based on their supposed existence are not necessarily true.

CONSISTENCY OF INDEPENDENT COUNTINGS AS A CRITERION FOR COMPLETENESS

By Thornton C. Fry*

1. **Origin of the Problem.** The statistical problem here considered arose in connection with the proof-reading of a set of mathematical tables, and for the sake of clarity can perhaps best be expressed in the language of that problem, although the results are obviously applicable to any situation in which a number of independent mechanisms (observers) are exposed to the same sequence of stimuli and are supposed to react to (count) them. It is conceivable that the results may be of value in many fields of biology and biophysics, as well as in connection with the reliability of the data of gangs of counting relays when used under conditions where failures of individual members of the gang may be expected to be frequent.

The proof-sheets of a set of mathematical tables were compared with copy by eight different proof-readers. As a group they located 34 *errors (hereafter referred to as the "known errors"). Individually, however, they missed* 14, 13, 12, 10, 7, 6, 4 *and* 4 *of these known errors respectively. What degree of confidence is warranted in the assertion that the proof does not contain "unknown" errors which all the observers missed?*

The question is evidently one of inverse probabilities and may therefore conceivably be solved either by means of Bayes' theorem or by a consistency criterion of the kind commonly known as "goodness of fit." The former of these methods would have the merit of giving an unqualified answer; that is, it would result in a statement of the form "The probability that an undiscovered error exists is so-and-so." But it requires a preliminary estimate of certain existence probabilities, the estimation of which in the present instance would be mere uneducated guesswork. By contrast a pronouncement of the consistency type is much less direct. It reads in the form "The probability of an undiscovered error does not exceed so-and-so unless the consistency of the records of the various proof-readers is a most unusual accident."

Since in the present instance the less direct pronouncement appears to be satisfactory, the argument will be framed along this line. Spe-

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cifically, we shall do the following things: We shall assume a set of values p_i which are tentatively supposed to measure the individual chances of the various readers missing a particular error. On the basis of this assumption we compute two probabilities:

(1) The probability that as a group the m readers will discover N and miss n of the $N + n$ errors. For brevity we call this "the probability of event A" and denote it by $P(A)$.

(2) The probability that, if the event A occurs, the findings of the various readers would be distributed in a manner less likely than that which was observed. This we call "the probability of event B" and denote it by $P_A(B)$.

With these two probabilities known as functions of $[p_i]$ we seek for that set $[p_i]$ which will make $P(A)$ as large as possible (say $\bar{M}$) subject to the limitation that $P_A(B)$ shall not fall below a preassigned magnitude $\underline{M}$. We shall then be in a position to make the following assertion:

If there are n undiscovered errors, no possible assumption regarding $[p_i]$ can make the probability of the event A exceed $\bar{M}$, without at the same time making the probability of event B less than $\underline{M}$.

2. **Mathematical Formulation of the Problem.** We begin by viewing the problem as it would appear before the proofs were read and thus, divested of any foreknowledge of its result, lay down as our fundamental postulate that there is a certain well-defined though unknown probability p_j that the j'th man will miss an error, and a probability $q_j = 1 - p_j$ that he will find it.[1] As the men read independently it follows that the probability that all m of them will miss a certain error is

$$r = \prod_{j=1}^{m} p_j ; \tag{1}$$

and the probability that they will discover N and miss n of any specified group of $N + n$ errors is

$$P(N, n; p_j) = C_n^{N+n} r^n (1 - r)^N. \tag{2}$$

This is what we have called above "the probability of event A."

Either (1) or (2) might be used in estimating in advance the confi-

[1] We do not contemplate the possibility that the errors belong to classes which differ widely in difficulty of detection—a problem with which little of value could be done on the basis of the meager evidence available, and which is perhaps of greater academic than practical importance.

dence which would be warranted in the results of our proof-reading if we knew the individual probabilities $[p_j]$, which of course we do not.

Next we raise the question, also viewed as asked before the proof is read, "What is the probability that the individual results of the m readings will be so distributed that (a) a preassigned group of n errors are missed by everyone; (b) the j'th man misses a_j and finds b_j of the remaining N; (c) none of these N is missed by everyone?" The probability of this we denote by $P(N, n; a_j; p_j)$. It is readily found to be[2]

$$P(N, n; a_j; p_j) = \sum_{k=0}^{a_m} (-1)^k C_k^N \prod_{j=1}^{m} C_{b_j}^{N-k} p_j^{a_j+n} q_j^{b_j}. \tag{3}$$

If N is fairly large and p_j/q_j reasonably small, this can be replaced by[3]

$$P(N, n; a_j; p_j) = \frac{e^{-\frac{1}{2}\sum y_j^2}}{\sigma_1\sigma_2 \cdots \sigma_m} \sum_{k=0}^{a_m} (-1)^k C_k^N (e^k/2\pi)^{m/2} r^{n+k}, \tag{4}$$

where $$y_j = \frac{a_j - Np_j}{\sigma_j}, \qquad \sigma_j^2 = Np_j(1 - p_j).$$

We next fix our attention on a particular set of values $[A_j]$ of the $[a_j]$. (This set will eventually be identified with the results obtained by the proof-readers.) Viewed from our a priori point of view there is a certain probability of the experiment leading to this set of a_j's. Equation (4) can, therefore, be used to classify all possible sets $[a_j]$ into two classes according as the probability of their occurrence does or does not exceed that of $[A_j]$. We now ask for the probability of a result belonging to the second class: that is, the probability that our experiment, viewed as still to be performed, will result in the discovery of the N errors and the failure to discover n, but with the set of individual results $[a_j]$ less probable than $[A_j]$. The result is[4]

$$P(N, n; A_j; p_j) = \frac{r^n P_m(> \chi^2)}{s} \sum_{k=0}^{a_m} C_k^N (-e^{m/2} r)^k.$$

To find the equivalent probability, subject to the condition that *some* N (not some *particular* N) out of $N + n$ errors are known to have been found, we must multiply by C_n^{N+n} and divide by (2). This gives

$$K(r, s, \chi^2) = \frac{P_m(> \chi^2)}{s(1 - r)^N} \sum_{k=0}^{a_m} C_k^N (- e^{m/2} r)^k, \tag{5}$$

[2] See Appendix for derivation. a_m is the smallest of $[a_j]$.
[3] Fry, "Probability and Its Engineering Uses," pp. 280–283.
[4] Fry, "Probability and Its Engineering Uses," pp. 285–289.

which, as the notation indicates, depends on the variables $[p_j]$ only through the functions[5]

$$r = \prod_{j=1}^{j=m} p_j, \tag{1}$$

$$s = \prod_{j=1}^{j=m} \sqrt{Np_j(1 - p_j)}, \tag{6}$$

$$\chi^2 = \sum_{j=1}^{j=m} \frac{(A_j - Np_j)^2}{Np_j(1 - p_j)}. \tag{7}$$

Obviously equation (5) gives the probability which we refer to above as "the probability of event B."

We have now completed that part of our study which involves the concept of probability. From this point on our purpose will be to find how large the probability (2) can be made subject to the limitation that (5) does not fall below some preassigned limit. As it is not difficult to show that the maximum value of (2) is a monotonic decreasing function of the limit thus assigned, we can throw the algebraic statement of the problem into a somewhat simpler form: namely, that we seek the maximum value of (2) subject to the limitation that (5) shall be a known constant.

For algebraic convenience in treating this problem we introduce the notation $F(r) \equiv P(N, n; p_j)$ to indicate that the probability (2) depends upon the variables $[p_j]$ only through r. Our problem is then to maximize the function $F(r) - \lambda K(r, s, \chi^2)$ with respect to the variables $[p_j]$. This requires that

$$\left[\frac{dF}{dr} - \lambda \frac{\partial K}{\partial r}\right] \frac{r}{p_j} + \lambda K \frac{1 - 2p_j}{2p_j(1 - p_j)} - \lambda \frac{\partial K}{\partial \chi^2} \frac{\partial \chi^2}{\partial p_j} = 0. \tag{8}$$

Multiplying this equation by p_j and subtracting it from the equivalent equation in p_m, we readily arrive at the set of $m - 1$ equations

$$\frac{1 - 2p_j}{2 - 2p_j} - p_j \frac{P_m'(> \chi^2)}{P_m(> \chi^2)} \frac{\partial \chi^2}{\partial p_j} = \frac{1 - 2p_m}{2 - 2p_m} - p_m \frac{P_m'(> \chi^2)}{P_m(> \chi^2)} \frac{\partial \chi^2}{\partial p_m}, \tag{9}$$

which must be satisfied by $[p_j]$ if our maximum is to be attained.

3. **Method of Numerical Calculation.** In developing a process of numerical calculation we first assign any convenient value to the

[5] It will also be noted, though the matter is of no consequence in the present connection, that k depends on the variables A_j only through χ^2.

function $P_m(> \chi^2)$. From its tabulated values[6] we can readily obtain the corresponding value of χ^2 and hence also the value of

$$P_m'(> \chi^2) = \frac{-\chi^{m-2}\, e^{-\chi^2/2}}{2^{m/2}\left(\frac{m}{2} - 1\right)!}.$$

Thus we obtain a value of $P_m'(> \chi^2)/P_m(> \chi^2)$, which we may denote by q. Using this value and the derivative $\frac{\partial \chi^2}{\partial p_j}$, as obtained from (7), the left-hand side of (9) is found to be

$$f(p_j) = \frac{1 - 2p_j}{2 - 2p_j} + \frac{q}{N}\frac{(A_j - Np_j)(A_j + Np_j - 2A_j p_j)}{p_j(1 - p_j)^2}, \qquad (10)$$

in which all quantities are known except the single variable p_j. A set of m curves can therefore be drawn to represent the m functions $f(p_1)$, $f(p_2), \cdots, f(p_m)$. But the equations (9) now take the form

$$f(p_1) = f(p_2) = \cdots = f(p_m); \qquad (11)$$

hence the abscissae obtained by taking one and the same ordinate on all these curves constitute a solution of (9). They do not however necessarily satisfy (7); for it must be remembered that the value of χ^2 has already been specified. But by choosing various ordinates in succession and computing the corresponding values of the right-hand member of (7) we may eventually find that one for which the equation is satisfied.

By this process we arrive at a set $[p_j]$ which makes the probability (2) as large as possible for some fixed value of (5), and it is obvious that the maximum value and fixed value in question are easily calculated once $[p_j]$ is known. In practice almost any pair of results which are reasonably small may be regarded as an acceptable solution of our problem; that is, the maximum value of F for any sensible value of K is normally answer enough. In stating our problem, however, we have placed the matter somewhat differently in that we have required the maximum of F for a *preassigned* K. Even this may be obtained if desired by a further process of cut-and-try; namely, by beginning with various values of $P(> \chi^2)$, computing through to K as explained above for each of them, and by interpolation from the set of values thus obtained, determining that set $[p_j]$ which yields the desired K.

[6] E.g., "Probability and Its Engineering Uses," Table VIII.

4. **Numerical Results.** We return now to the specific numerical problem presented in Section 1. Computations were made using $N = 34$ and $n = 1$ and starting with the four values 0.001, 0.1, 0.9 and 0.99 for $P(> \chi^2)$. The results are shown below.

$P(> \chi^2)$	$\overline{M} = P(34, 1; p_i)$	$M = K(r, s, \chi^2)$
0.001	0.020	8.8×10^{-6}
0.1	0.007	2.3×10^{-5}
0.9	0.002	4.1×10^{-4}
0.99	0.001	5.5×10^{-4}

The last entry is perhaps the only one of interest, the others being the results of "false starts." It states that: no matter what the $[p_i]$ may have been, it must either be true that the probability of having overlooked an error is less than 1 in 1000, or else that the chance of getting results as bizarre as the proof-readers actually did is itself less than 1 in 1800.

A high degree of confidence in the accuracy of the tables would therefore appear warranted.

The amount of time required to secure a result is of importance in appraising the practical merits of any cut-and-try solution. It may, therefore, be of interest to state that the above computations required about twelve hours, using an electrical computing machine and a skilled operator.

Appendix

We shall derive formula (3) from a difference equation which, in effect, expresses a relationship among the values taken by (3) for various values of n. To do this we shall assume the group N divided into two subgroups of μ and ν errors, respectively, as shown schematically in the diagram. It will be helpful to regard the subgroup ν as a hypothetical enlargement of the group of undiscovered errors. As this enlargement is made at the expense of the group which we have characterized as "discovered," it of course violates the specific conditions of our problem. In effect it introduces a class of similar problems in which the rôles of N and n are taken by μ and $n + \nu$, respectively.

We also introduce two closely related probabilities which we may denote for the time being by $P(\mu)$ and $P^*(\mu)$, respectively. $P(\mu)$ is the exact analog of (3) in our extended class of problems: that is, it is the chance of all $n + \nu$ being missed by everyone, b_i of the μ being found

by the j'th man, all of the μ found by someone. $P^*(\mu)$ differs only in that the last condition is not implied, so that distributions $b_1, b_2, \cdots, b_m$ will be accounted favorable whenever each is contained in μ, even though in the aggregate they do not contain μ.

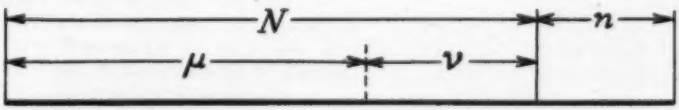
Obviously, then, the situation contemplated by $P^*(\mu)$ includes the following possibilities:

Either no member of the group μ is overlooked, the probability of which is $P(\mu)$;

Or else some one of the group μ is overlooked, the probability of which is $C_1^n P(\mu - 1)$;

Or else two of the group μ are overlooked, the probability of which is $C_2^n P(\mu - 2)$;

Or else $\cdots$.



Assembling these results we obtain the equation

$$P^*(\mu) = P(\mu) + C_1^n P(\mu - 1) + C_2^n P(\mu - 2) + \cdots + C_{\mu - b_m}^n P(b_m), \quad \text{(A)}$$

where b_m is assumed to be the largest of the b's.

However, it is also easily seen that

$$P^*(\mu) = \prod_{j=1}^{m} C_{b_j}^{a_j + n} p_j^{a_j + n} q_j^{b_j}. \quad \text{(B)}$$

Furthermore the distinction between $P(\mu)$ and $P^*(\mu)$ becomes trivial when $\mu = b_m$. Hence

$$P(b_m) = P^*(b_m). \quad \text{(C)}$$

If now we write down the set of equations (A) for all values of μ from b_m to N and solve for $P(N)$ we get, with the aid of (B) and (C),

$$P(N) = \sum_{k=0}^{a_m} (-1)^k C_k^N \prod_{j=1}^{m} C_{b_j}^{N-k} p_j^{a_j + n} q_j^{b_j}. \quad \text{(D)}$$

Obviously, however, in the simplified notation here used $P(N)$ has the same meaning as the $P(N, n; a_j; p_j)$ of the text. Hence (3) is verified.

ON CERTAIN THREE-DIMENSIONAL MANIFOLDS IN EUCLIDEAN SIXSPACE (V_3 IN R_6)

By Nathan Kaplan*

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Introduction

The general theory of m-dimensional manifolds (V_m) with a Riemannian, positive definite metric ($a'_{\lambda\kappa}$) which are imbedded in n-dimensional ordinary Euclidean space (R_n) has been studied by Ricci, Kühne, Schouten, Struik, Eisenhart, Burstin, Mayer and others. Schur, Bianchi, Banal and Cartan have investigated the theory of deformable manifolds; while Segre, Bompiani, Togliatti, Terracini and others have contributed to the projective theory of such manifolds.

In this paper, we shall offer: (1) a method of studying V_m in an R_n; (2) a study of a class of V_3 in an R_6 which are immediate generalizations of developable V_2 in an R_3.

I desire to thank Prof. D. J. Struik for the considerable time which

* In part, a thesis for the doctorate, accepted by the Massachusetts Institute of Technology.

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he has spent with me in supervising my thesis, contained in the first six sections of this work. Prof. Struik has also offered helpful suggestions in connection with the latter part of this paper.

I. Notation and Method

A. *Notation.* We shall use the general notation of Schouten-Struik.[1] In ordinary Euclidean sixspace (R_6), that is a six space with positive definite fundamental tensor, we use the Euclidean orthogonal coördinates denoted by:

$$y^\kappa, \qquad (\kappa, \lambda \cdots = 1 \cdots 6). \tag{1}$$

In the R_6, we introduce a three dimensional manifold by the equations

$$y^\kappa = y^\kappa(u^a), \qquad (a, b = 1, 2, 3). \tag{2}$$

If the tangent three-spaces of this manifold have no special position with respect to the cone of zero directions, a Riemannian metric is induced in this manifold and it may accordingly be denoted by V_3. The (u^a) form a system of intrinsic coördinates of the V_3.

For the unit tangent vector field of congruences of V_3, we write

$$\underset{c}{i^\lambda} \equiv i^\lambda \qquad (c, d \cdots = 1, 2, 3); \tag{3}$$

for the unit tangent vector field of congruences in the normal V_6^{6-3}

$$\underset{p}{i^\kappa} \equiv i^\kappa, \qquad (p, q \cdots = 4, 5, 6). \tag{4}$$

For the first fundamental tensor of the V_3 we write

$$a'_{\lambda\kappa}, \qquad (\lambda, \kappa, \cdots = 1 \cdots 6). \tag{5}$$

For the unit affinor of the V_3, we use

$$B^\kappa_\lambda = \sum_{c=1}^{3} i^\kappa_c i_\lambda, \tag{6}$$

for the second fundamental tensors,

$$\underset{p}{h_{\lambda\lambda}} \equiv h_{\lambda\lambda} = B^{\beta\alpha}_{\lambda\lambda} \nabla_\beta \underset{p}{i_\alpha}, \tag{7}$$

for the Riemann-Christofel affinor,

$$K'_{dcba} = B^{\nu\mu\lambda\kappa}_{dcba} K_{\nu\mu\lambda\kappa}. \tag{8}$$

and for the vectors entering into the Codazzi relations,

$$(9) \qquad \overset{pq}{v}_\lambda \equiv \underset{p}{\overset{q}{v}}_\lambda = B^\alpha_\lambda (\nabla_\alpha \underset{p}{i_\beta}) \underset{q}{i^\beta}, \qquad (p, q = 4, 5, 6).$$

B. *Method.* From a theorem of Eisenhart:[3] "Given an R_6 and the quantities

$$(10) \qquad \text{(a)} \quad \underset{p}{\overset{q}{v}}_\lambda, \qquad \text{(b)} \quad \underset{p}{\overset{p}{h}}_{\lambda\alpha}, \qquad \text{(c)} \quad a'_{\lambda\alpha},$$

satisfying the following equations known as the Gauss, Codazzi and Ricci relations respectively,[3]

$$(11) \qquad \text{(a)} \quad K'_{\nu\mu\lambda\alpha} = \left(\underset{p}{\overset{p}{h}}_{\nu\lambda} \underset{p}{\overset{p}{h}}_{\mu\alpha} - \underset{p}{\overset{p}{h}}_{\nu\alpha} \underset{p}{\overset{p}{h}}_{\mu\lambda} \right),$$

$$\text{(b)} \quad \nabla'_{[\lambda} \underset{p}{h}_{\mu]\lambda} = \left(\underset{p}{\overset{q}{v}}_{[\nu} \underset{q}{h}_{\mu]\lambda} \right),$$

$$\text{(c)} \quad \nabla'_{[\nu} \underset{p}{\overset{q}{v}}_{\lambda]} = \left(\underset{p}{\overset{r}{v}}_{[\nu} \underset{r}{\overset{q}{v}}_{\lambda]} + \underset{p}{\overset{q}{h}}^{\sigma}_{[\nu} \underset{p}{h}_{\lambda]\sigma} \right),$$

then a V_3 in an R_6 is determined." Our method of analysis depends upon a classification of (10, *a*, *b*) and the determination of $a'_{\lambda\alpha}$ consistent with (11).

The geometric meaning of this type of classification will now be considered. The $(h_{\lambda\alpha})$ offer no difficulty as their meanings are known. By (9), we see that when any set of mutually orthogonal V_6^{6-3} vectors $\underset{p}{(i_\lambda)}$ are given then $\overset{pq}{(v_\lambda)}$ are determined. We shall show the converse is valid—namely, each set of $\underset{p}{\overset{q}{(v_\lambda)}}$ corresponds to a unique set of $\underset{p}{(i_\lambda)}$. In order to do this, we study rotations in the space of normals V_6^{6-3}.

Under a rotation any set of $\underset{p}{(i_\lambda)}$ is transformed by the orthogonal transformation $\underset{p}{\overset{q}{[\alpha(u^\alpha)]}}$,

$$(12) \qquad \underset{p}{{}'i_\lambda} = \underset{p}{\overset{q}{\alpha}}\, \underset{q}{i_\lambda}.$$

From (9) we derive the equation:

$$'\overset{q}{\underset{p}{v}}_{\lambda} = B^{\kappa}_{\lambda}\left(\nabla_{\kappa}\,'\overset{}{\underset{p}{i}}_{\mu}\right)'i^{\mu}, \tag{13}$$

$$= \sum_{r}\overset{r}{\underset{q}{\alpha}}\nabla'_{\lambda}\overset{r}{\underset{p}{\alpha}} + \sum_{s}\overset{s}{\underset{q}{\alpha}}\overset{r}{\underset{p}{\alpha}}\overset{s}{\underset{r}{v}}_{\lambda}.$$

Upon multiplying both sides of (13) by $\overset{t}{\underset{q}{\alpha}}$ and summing on (q),

$$\overset{t}{\underset{q}{\alpha}}\,'\overset{q}{\underset{p}{v}}_{\lambda} = \delta^{tr}\nabla'_{\lambda}\overset{r}{\underset{p}{\alpha}} + \delta^{ts}\overset{r}{\underset{p}{\alpha}}\overset{s}{\underset{r}{v}}_{\lambda}. \tag{14}$$

Or

$$\boxed{\nabla'_{\lambda}\overset{t}{\underset{p}{\alpha}} = \overset{t}{\underset{q}{\alpha}}\,'\overset{q}{\underset{p}{v}}_{\lambda} - \overset{r}{\underset{p}{\alpha}}\overset{t}{\underset{r}{v}}_{\lambda}}. \tag{15}$$

Let us briefly write the first integrability conditions of (15):

$$\nabla'_{[\mu}\nabla'_{\lambda]}\overset{t}{\underset{p}{\alpha}} = 0. \tag{16}$$

By means of (15) we can reduce these equations (16) to

$$\overset{r}{\underset{t}{\alpha}}\left(\nabla'_{[\mu}\,'\overset{t}{\underset{p}{v}}_{\lambda]} - \,'\overset{s}{\underset{p}{v}}_{[\mu}\,'\overset{t}{\underset{p}{v}}_{\lambda]}\right) = \overset{s}{\underset{p}{\alpha}}\left(\nabla'_{[\mu}\overset{r}{\underset{s}{v}}_{\lambda]} - \overset{t}{\underset{s}{v}}_{[\mu}\overset{r}{\underset{t}{v}}_{\lambda]}\right). \tag{17}$$

If we write

$$\overset{r}{\underset{p}{v}}_{\mu\lambda} = \nabla'_{[\mu}\overset{r}{\underset{p}{v}}_{\lambda]} - \overset{t}{\underset{p}{v}}_{[\mu}\overset{r}{\underset{t}{v}}_{\lambda]}, \tag{18}$$

we see that the C^{n}_{2} matrices $v_{\mu\lambda} = \left[\overset{r}{\underset{p}{v}}_{\mu\lambda}\right]$ are antisymmetric and (17) becomes,

$$\overset{r}{\underset{r}{\alpha}}\,'\overset{p}{\underset{p}{v}}_{\mu\lambda} = \overset{r}{\underset{s}{v}}_{\mu\lambda}\overset{s}{\underset{p}{\alpha}}, \tag{19}$$

or in matrix language:

$$\boxed{\alpha\,'v_{\mu\lambda} = v_{\mu\lambda}\alpha}. \tag{20}$$

THEOREM I. *Each set of* $\overset{q}{\underset{p}{v}}_{\lambda}$ *corresponds to a unique set of* $\underset{p}{i}_{\lambda}$.

For assume the contrary, then an *orthogonal matrix* α exists such that

$$\nabla'_\lambda \alpha = \alpha v_\lambda - v_\lambda \alpha, \tag{21}$$

$$\alpha v_{\mu\lambda} = v_{\mu\lambda} \alpha, \tag{22}$$

where v_λ is the matrix $\begin{bmatrix} i \\ v_\lambda \\ r \end{bmatrix}$ and

$$v_{\mu\lambda} = \nabla'_{[\mu} v_{\lambda]} - v_{[\mu} v_{\lambda]} . \tag{23}$$

Since the matrix (α) is symmetric, a real orthogonal transformation exists which reduces (α) to canonical form. Let (T) be this transformation and (β) the canonical form matrix. Then

$$T\alpha T^{-1} = \beta. \tag{24}$$

And using the equation (22),

$$\beta(T v_{\mu\lambda} T^{-1}) = (T v_{\mu\lambda} T^{-1})\beta. \tag{25}$$

Thus β is canonical and $(T v_{\mu\lambda} T^{-1})$ is antisymmetric. But if two matrices commute then they can both be reduced to canonical form simultaneously (same T). Thus since $(T v_{\mu\lambda} T^{-1})$ is antisymmetric and in canonical form

$$T v_{\mu\lambda} T^{-1} = 0. \tag{26}$$

By multiplication with (T^{-1}), (T), we conclude

$$v_{\mu\lambda} = 0. \tag{27}$$

Thus only if (27) is satisfied, can (22) exist. We shall call normal spaces for which $v_{\mu\lambda} = 0$ "*isotropic.*" Theorem I is valid for all normal spaces except isotropic normal spaces (note $n - m = 2, 3$, for V_n^{n-m}).

II. Some Imbedment Theorems

Before proceeding to a direct study of V_3 in an R_6, it is necessary that we determine, when V_3 in an R_6 degenerate into V_3 in an R_4 or V_3 in an R_5.

Theorem II. *The necessary and sufficient condition that any V_3 in an R_6 be a V_3 in an R_4 is that for some set of mutually orthogonal normal vectors* $\left(\underset{p}{i_\lambda}\right)$,

$$\underset{p}{v_\lambda}^{q} = 0, \qquad h_{\lambda\alpha} = h_{\lambda\alpha} = 0. \tag{1}$$

Consider a V_3 in an R_6 for which our condition is satisfied. In this case, the Gauss, Codazzi and Ricci relations (I, 11) reduce to those satisfied by a V_3 in an R_4. Hence by (I, 10), a V_3 in an R_4 is *actually determined.* Thus our V_3 lies in an R_4. Conversely, if our V_3 in an R_6 is a V_3 in an R_4, select the normals $\left(\overset{q}{i}_{\lambda}, \overset{\lambda}{i}_{5}\right)$ as perpendicular to the R_4.

It is evident from the definitions of $\overset{q}{v}_{\lambda}$ and $h_{\lambda\alpha}$ that

$$\overset{q}{v}_{\lambda} = 0, \qquad h_{\lambda\alpha} = h_{\lambda\alpha} = 0. \tag{2}$$

Q.E.D.

THEOREM III. *The necessary and sufficient condition that a V_3 in an R_6 be a V_3 in an R_5 is that for some set of mutually orthogonal normal vectors* $\left(\overset{\lambda}{i}\right)$,

$$\overset{6}{v}_{\lambda} = \overset{6}{v}_{\lambda} = 0, \qquad h_{\lambda\alpha} = 0. \tag{3}$$

The proof is exactly the same as the previous theorem.

III. V_3 IN AN R_6 OF [1, 1]

We shall study V_3 in an R_6 whose normal spaces are isotropic and whose (3) second fundamental tensors are of rank (1). These V_3 in the R_6 shall be denoted by [1, 1]; the first index denoting the classification of $\overset{q}{v}_{\lambda}$; the second index, the classification of $h_{\lambda\alpha}$. Hence

$$\overset{q}{v}_{\lambda} = 0; \qquad h_{\lambda\alpha} = h\, i_{\mu}\, i_{\lambda}\,; \qquad h_{\lambda\alpha} = h\, i_{\lambda}\, i_{\alpha}\,; \qquad h_{\lambda\alpha} = h\, i_{\lambda}\, i_{\alpha}\,. \tag{1}$$

The Gauss, Codazzi and Ricci relations become

$$K'_{dcba} = 0, \tag{2}$$

$$\nabla'_{[a} h_{\mu]\lambda} = 0, \tag{3}$$

$$h'_{[\mu} h_{\lambda]\rho} = 0. \tag{4}$$

From (2), we see that V_3 are "applicable" to R_3. Substituting (1) into (4), we find

$$\left(i^{\lambda}_{a}\, i_{\lambda b}\right)\left(i_{[\mu a}\, i_{\alpha]b}\right) = 0, \qquad (a, b = 1, 2, 3). \tag{5}$$

There are C_2^3 equations and all solutions to these equations can be obtained by placing either of the above factors equal to zero. Placing the first factor equal to zero, we find that $\left(\underset{a}{i_\lambda}\right)$ is perpendicular to $\left(\underset{b}{i_\lambda}\right)$. Placing the second factor equal to zero, we find that $\left(\underset{b}{i_\lambda}\right)$ is in the direction of $\left(\underset{a}{i_\lambda}\right)$. Thus we have three types of solution,

(6) $$\underset{1}{i_\lambda} = \underset{2}{i_\lambda} = \underset{3}{i_\lambda};$$

(7) $$\underset{1}{i_\lambda} = \underset{2}{i_\lambda}, \ \underset{3}{i_\lambda} \text{ is orthogonal to } \underset{2}{i_\lambda};$$

(8) $$\underset{1}{i_\lambda}, \ \underset{2}{i_\lambda}, \ \underset{3}{i_\lambda} \text{ are mutually orthogonal.}$$

We shall denote the V_3 which satisfy (1), (6) by $[1, 1]_a$; those which satisfy (1), (7) by $[1, 1]_b$; and those which satisfy (1), (8) by $[1, 1]_c$.

IV. V_3 IN R_6 OF $[1, 1]_a$

The V_3 of this class satisfy the conditions

(1) $$\overset{q}{\underset{p}{v_\lambda}} = 0,$$

(2) $$\underset{4}{h_{\mu\lambda}} = \underset{4}{h}\, i_\mu\, \underset{1}{i_\lambda},$$

(3) $$\underset{5}{h_{\mu\lambda}} = \underset{5}{h}\, i_\mu\, \underset{1}{i_\lambda},$$

(4) $$\underset{6}{h_{\mu\lambda}} = \underset{6}{h}\, i_\mu\, \underset{1}{i_\lambda}.$$

We must determine whether these V_3 in an R_6 are V_3 in an R_4 or R_5. Under the orthogonal transformation (I, 12), the $\underset{p}{h_{\mu\lambda}}$ transform as follows:

(5) $$\underset{p}{'h_{\mu\lambda}} = \overset{q}{\underset{p\ q}{\alpha}}\, \underset{q}{h_{\mu\lambda}}.$$

Now let us assume that these V_3 lie in an R_4. Then by Theorem II, a set of $\left(\underset{p}{'i_\lambda}\right)$ exist for which

(6) $$\overset{q}{\underset{p}{'v_\lambda}} = 0, \qquad \underset{4}{'h_{\mu\lambda}} = \underset{5}{'h_{\mu\lambda}} = 0.$$

Thus by (I, 15)

(7) $$\nabla'_\lambda \overset{q}{\underset{p}{\alpha}} = 0.$$

The only solution of this system in terms of surface variables (u^α) is

$$\overset{u}{\underset{p}{\alpha}} = \text{constant}. \tag{8}$$

Upon substitution of (6) and (2), (3), (4) into (5), we obtain

$$\overset{q}{\underset{4\ q}{\alpha}} h = 0, \tag{9}$$

$$\overset{q}{\underset{5\ q}{\alpha}} h = 0. \tag{10}$$

Hence conditions (8), (9), (10) must be satisfied by the $\overset{q}{h}$ in order that V_3 lie in an R_4. By use of Theorem III, we can similarly show that these V_3 lie in an R_5 only when

$$\overset{q}{\underset{6}{\alpha}} = c, \qquad q = 4, 5, 6, \tag{11}$$

$$\overset{q}{\underset{6\ q}{\alpha}} h = 0. \tag{12}$$

We shall show in the future that $\overset{q}{h}$ can always be chosen so as to satisfy the Gauss, Codazzi, and Ricci relations yet not (9, 10) nor (11, 12).

We return to the general analysis of $[1, 1]_a$. For the curvature vector, we find, where (i^λ) is an arbitrary unit vector in the tangent R_3 at each point P,

$$''u^\alpha = \left(\underset{p}{h_{\mu\lambda}}\, i^\mu i^\lambda\right) \overset{p}{i^\alpha}, \tag{13}$$

$$= \left(\underset{1}{i^\lambda i_\lambda}\right)^2 \left(\overset{q}{h}\, i^\alpha\right).$$

For asymptotic directions $\left(\underset{a}{i^\lambda}\right)$,

$$\underset{a\ 1}{i^\lambda i_\lambda} = 0. \tag{14}$$

For conjugate directions $\left(\underset{b}{i^\alpha}, \underset{c}{i^\alpha}\right)$,

$$''u^\alpha_{bc} = \left(\underset{p}{h_{\mu\lambda}}\, \underset{b}{i^\mu}\, \underset{c}{i^\lambda}\right) \overset{p}{i^\alpha} = 0, \tag{15}$$

$$\left(\underset{b\ 1}{i^\lambda i_\lambda}\right) = 0, \qquad \left(\underset{c\ 1}{i^\lambda i_\lambda}\right) = 0. \tag{16}$$

We can summarize these results by the following description of the curvature properties of the V_3 : the V_3 have axial points; the symmetric cones of directions about $\left(\underset{1}{i_\lambda}\right)$ have the same normal curvature; at any point P in the V_3, the R_2 perpendicular to $\left(\underset{1}{i^\lambda}\right)$ contains all the asymptotic directions; all directions in this R_2 are asymptotic; any direction in this R_2 is conjugate to any other direction in the tangent R_3 ; no other conjugate directions exist.

In order to analyze the Codazzi relations (I, 11, b), we use the formulae of Struik[4]

$$\underset{b}{i_\lambda}\,\underset{1}{i^\lambda} = 0, \qquad (b, c = 2, 3), \tag{17}$$

$$\left(\nabla'_\mu \underset{b}{i_\lambda}\right)\underset{1}{i^\lambda}\,\underset{c}{i^\mu} = 0, \tag{18}$$

$$\underset{b}{i^\mu}\,\underset{c}{i^\lambda}\,\nabla'_{[\mu}\underset{1}{i_{\lambda]}} = 0, \tag{19}$$

$$\underset{b}{i^\mu}\,\underset{b}{i^\lambda}\,\nabla'_\mu \underset{1}{i_\lambda} = 0, \tag{20}$$

$$\underset{b}{i^\mu}\,\nabla'_\mu \underset{p}{b} + h\,\underset{p\,1}{i^\mu}\,\underset{1}{i^\lambda}\,\nabla'_\mu \underset{b}{i_\lambda} = 0. \tag{21}$$

These results may be summarized as follows: $\underset{2}{i_\lambda}$, $\underset{3}{i_\lambda}$ are any two mutually orthogonal vectors in the R_2 perpendicular to $\underset{1}{i_\lambda}$; these vector-fields $\left(\underset{2}{i_\lambda}, \underset{3}{i_\lambda}\right)$ are V_2-normal; $\left(\underset{1}{i_\lambda}\right)$ is V_2-normal, from (20), we see that the V_2-component of normal curvature of any vector in the R_2 is zero. This last means that the V_2 built by the vector fields $\left(\underset{2}{i_\lambda}, \underset{3}{i_\lambda}\right)$ are geodesic in the V_3. Since any vector $\left(\underset{b}{i_\lambda}\right)$ lies in the nulldomain of $\left(\underset{p}{h_{\mu\lambda}}\right)$, its components of normal curvature along $\left(\underset{p}{i_\lambda}\right)$ are zero. But the only V_2 in an R_6 which have zero normal curvature are flat R_2's or planes. Thus, the V_3 consist of ∞^1 planes and admit ∞^2 triply orthogonal systems of V_2 .

If we write our linear element in terms of some triply orthogonal net in the V_3 as

$$ds^2 = a'_{11}(du^1)^2 + a'_{22}(du^2)^2 + a'_{33}(du^3)^2, \tag{22}$$

$$ds^2 \equiv a'_{cb}\,du^c\,du^b, \qquad a'_{22} = a'_{33} = 1,$$

then the equations (21) become

$$\frac{\partial h}{\partial u^2} = -\frac{h}{2a'_{11}}\frac{\partial a'_{11}}{\partial u^2}, \tag{23}$$

$$\frac{\partial h}{\partial u^3} = -\frac{h}{2a'_{11}}\frac{\partial a'_{11}}{\partial u^3}. \tag{24}$$

Upon integration, these equations become

$$\ln h = \ln (a'_{11})^{-\frac{1}{2}} + f_p(u^1). \tag{25}$$

Or in other words,

$$h = \frac{g_r(u^1)}{(a'_{11})^{\frac{1}{2}}}, \tag{26}$$

where the functions $g_p(u^1)$ are arbitrary. Since h must satisfy only this condition in order that the Codazzi and Ricci relations be satisfied and since the Gauss relations impose no conditions on h (as we shall show), then conditions of the form (9, 10) or (11, 12) are not satisfied. The theorem which summarizes these results is:

THEOREM IV. *If and only if the $g_p(u^1)$ satisfy two linear relations with constant coefficients*

$$\alpha g_p = 0, \qquad \alpha g_p = 0, \qquad \sum_{p=4}^{6} \alpha \alpha = 0 \tag{27}$$

does the V_3 lie in an R_4.

If and only if the $g_p(u^1)$ satisfy one linear relation with constant coefficients

$$\alpha g_p = 0, \tag{28}$$

does the V_3 lie in an R_5.

We can give an interesting geometric interpretation of (9, 10) and (11, 12). Let $\left(h, h, h\right)$ denote the components of a vector in the normal V_6^{6-3}. Then if this vector has a fixed direction with respect to $\left(i_\lambda\right)$ the V_3 lies in an R_4; if this vector has a two dimensional domain (R_2) then the V_3 lies in an R_5; if at any three points of V_3, this vector assumes three independent positions $\left(\text{with respect to } i_\lambda\right)$ then V_3 lies in an R_6.

The Gauss relation (I, 11, a) becomes

$$K'_{rsls} = 0. \tag{29}$$

In order to complete our direct study of the V_3 of $[1, 1]_a$, we must show that a linear element of the type (22), which is *equivalent to the properties* (17) *to* (20), is a solution of (29). Using the conditions (22) and the formulae of Eisenhart,[5] the system (29) reduces to

$$\frac{\partial^2 a}{\partial u^{\overset{1}{2}\,\overset{1}{2}}} = 0, \qquad \frac{\partial^2 a}{\partial u^{\overset{1}{3}\,\overset{1}{3}}} = 0, \qquad \frac{\partial^2 a}{\partial u^{\overset{1}{2}}\partial u^{\overset{1}{3}}} = 0. \tag{30}$$

Where

$$a_1 = \sqrt{a'_{11}}, \qquad a'_{22} = a'_{33} = 1. \tag{31}$$

A solution of the system (30) is given by

$$a_1 = 1 + u^2\theta_2(u^1) + u^3\theta_3(u^1), \tag{32}$$

where θ_2, θ_3 are arbitrary functions of the variable (u').

To summarize, if we take

$$a'_{bc} = 0, \quad b \neq c); \qquad (b, c = 1, 2, 3), \tag{33}$$

$$a'_{11} = [1 + u^2\theta_2(u^1) + u^3\theta_3(u^1)]^2, \tag{34}$$

$$a'_{22} = a'_{33} = 1, \tag{35}$$

$$\underset{p}{h}_{11} = \underset{p}{h}\, a'_{11}, \qquad (p, q = 4, 5, 6), \tag{36}$$

$$\underset{p}{h}_{22} = \underset{p}{h}_{33} = \underset{p}{h}, \tag{37}$$

$$\underset{p}{h} = g_p(u^1)(a'_{11})^{-\frac{1}{2}}, \tag{38}$$

$$\underset{p}{h}_{bc} = 0, \qquad (b \neq c), \tag{39}$$

$$\underset{p}{v}_c^{\,q} = 0, \tag{40}$$

where g_p, θ_2, θ_3 are arbitrary, then the Gauss, Codazzi and Ricci relations (written in V_3 coördinates) are satisfied and a V_3 in an R_6 of type $[1, 1]_a$ is determined.

We have shown that these V_3 contain ∞^1 planes (R_3) and these V_3 are applicable to an R_3. By bending these V_3 into R_3, the ∞^1 planes

go into planes and thus intersect in ∞^1 R_1 (lines). These lines are of course the tangents to a curve V_1''; the ∞^1 planes are the osculating planes of this V_1''. Hence the V_3 is the envelope of the ∞^1 osculating R_3 to the curve V_1''.

We shall show in the next section that the ∞^1 planes perpendicular to an arbitrary V_1 in an R_4 and such that two adjacent planes intersect in an R_1 (line), are of the type $[1, 1]_4$. Of course the V_1 can be drawn thru a given ∞^1 R_2 such that adjacent R_2 intersect.

V. V_3 in R_4 Generated by ∞^1 Planes Intersecting in ∞^1 R_1

Let us consider a curve V_1 in an R_4,

$$x^\lambda = x^\lambda(u^1), \tag{1}$$

$$\frac{\partial x^\lambda}{\partial u^1} = \underset{1}{i}{}^\lambda(u^1), \tag{2}$$

since u^1 measures arc length along V_1. Let $\underset{2}{i}{}^\lambda(u^1)$, $\underset{3}{i}{}^\lambda(u^1)$ be two mutually orthogonal vectors in the ∞^1 R_2. Then we define the V_3 by

$$y^\varkappa = x^\varkappa + u^2 \underset{2}{i}{}^\varkappa + u^3 \underset{3}{i}{}^\varkappa, \tag{3}$$

$$\frac{\partial y^\varkappa}{\partial u^1} = \underset{1}{i}{}^\varkappa + u^2 \frac{\partial \underset{2}{i}{}^\varkappa}{\partial u^1} + u^3 \frac{\partial \underset{3}{i}{}^\varkappa}{\partial u^1}, \tag{4}$$

$$\frac{\partial y^\varkappa}{\partial u^2} = \underset{2}{i}{}^\varkappa, \qquad \frac{\partial y^\varkappa}{\partial u^3} = \underset{3}{i}{}^\varkappa. \tag{5}$$

As yet we have not completely determined the fields $\left(\underset{2}{i}{}^\varkappa, \underset{3}{i}{}^\varkappa\right)$. All that we have said is that they lie in an R_2. We fix the field $'\underset{2}{i}{}^\varkappa$ at the plane P' infinitesimally removed from the plane P such that $'\underset{2}{i}{}^\lambda$ is orthogonal to the field $\underset{3}{i}{}^\varkappa$ at plane P. This means

$$\sum_\varkappa \underset{3}{i}{}^\varkappa \frac{\partial \underset{2}{i}{}^\varkappa}{\partial u^1} = 0. \tag{6}$$

Now let us calculate the first fundamental tensor of the V_3.

$$(7)\quad a'_{11} = \sum_{\kappa} \frac{\partial y^{\kappa}}{\partial u^1}\frac{\partial y^{\kappa}}{\partial u^1},$$

$$= \left[1 + 2u^2\left(\sum_{\kappa} i_1^{\kappa}\frac{\partial i_2^{\kappa}}{\partial u^1}\right) + 2u^3\left(\sum_{\kappa} i_1^{\kappa}\frac{\partial i_3^{\kappa}}{\partial u^1}\right) + 2u^2u^3\left(\sum_{\kappa}\frac{\partial i_2^{\kappa}}{\partial u^1}\frac{\partial i_3^{\kappa}}{\partial u^1}\right)\right.$$

$$\left. + \overline{u^2}^2\left(\sum_{\kappa}\frac{\partial i_2^{\kappa}}{\partial u^1}\frac{\partial i_2^{\kappa}}{\partial u^1}\right) + \overline{u^3}^2\left(\sum_{\lambda}\frac{\partial i_3^{\kappa}}{\partial u^1}\frac{\partial i_3^{\kappa}}{\partial u^1}\right)\right],$$

$$(8)\quad a'_{12} = -\sum_{\lambda} i_3^{\kappa}\frac{\partial i_2^{\kappa}}{\partial u^1} = 0,$$

$$(9)\quad a'_{13} = \sum_{\lambda} i_3^{\kappa}\frac{\partial i_2^{\kappa}}{\partial u^1} = 0,$$

$$(10)\quad a'_{23} = 0,$$

$$(11)\quad a'_{22} = a'_{33} = 1.$$

In virtue of (6), we can write $\left(\underset{2}{\theta}, \underset{3}{\theta}\right.$ arbitrary functions of $\left.(u^1)\right)$,

$$(12)\qquad \frac{\partial i_2^{\kappa}}{\partial u^1} = \underset{2}{\theta}\, i_1^{\kappa},$$

$$(13)\qquad \frac{\partial i_3^{\kappa}}{\partial u^1} = \underset{3}{\theta}\, i_1^{\kappa}.$$

It should be noted that we could condition our fields $\left(i_2^{\lambda}, i_3^{\lambda}\right)$ by (6) because of the fact that successive planes intersect. The geometric interpretation of (12), (13) is the fact that successive planes intersect. Upon substituting (12), (13) into (7), the latter becomes

$$(14)\qquad a'_{11} = \left(1 + u^2\underset{2}{\theta} + u^3\underset{3}{\theta}\right)^2,$$

where $\underset{2}{\theta}(u^1)$, $\underset{3}{\theta}(u^1)$ are arbitrary functions. We note that (12), (13) give a geometric interpretation of the functions $\underset{2}{\theta}$, $\underset{3}{\theta}$ which also enter into (IV, 34). In fact, $\underset{2}{\theta}$ is the angle between successive i_2^{κ} vectors along the V_1; likewise $\underset{3}{\theta}$ is the angle between successive i_3^{λ} vectors along the V_1.

We shall next determine the $\left(\underset{p}{h_{bc}}\right)$ of these V_3 in an R_4. To do this, we use the generalized Weingarten formulae:

$$D_c B_b^{\lambda} = H_{cb}^{\cdot\cdot\lambda}. \tag{15}$$

Upon reduction to V_3 coördinates, this becomes:

$$\frac{\partial^2 y^s}{\partial u^c \partial u^b} - \Gamma_{cb}^{a} \frac{\partial y^s}{\partial u^a} = -\underset{p}{h_{cb}}\, \overset{p}{i}{}^s. \tag{16}$$

We shall consider all of these formulae except those for which $c = b = 1$. By means of (3, 4, 5) of this section and some formulae of Eisenhart,[6] the system (16) becomes

$$\underset{p}{h_{cb}}\, \overset{p}{i}{}^s = 0, \qquad (c = b \neq 1). \tag{17}$$

Thus

$$\underset{p}{h_{cb}} = 0, \tag{18}$$

for all c, b except $c = b = 1$.

Upon expanding $\underset{p}{h_{cb}}$ in terms of the orthogonal 3-nuple $\left(\underset{1}{i}{}^s, \underset{2}{i}{}^s, \underset{3}{i}{}^s\right)$ and using (18), we find

$$\underset{p}{h_{cb}} = \underset{p\,1\,1}{h}\, i_c\, i_b. \tag{19}$$

We shall finally show that $\underset{p}{v_c}{}^{q} = 0$ for these V_3. From the equations of (19), it follows that

$$\underset{q}{h^{e}}{}_{\cdot[e}\underset{p}{h_{b]e}} = 0. \tag{20}$$

And the Ricci relations in conjunction with these give

$$\nabla'_{[c}\underset{p}{v}{}^{q}_{b]} - \underset{p}{v}{}^{r}_{[c}\underset{r}{v}{}^{q}_{b]} = 0. \tag{21}$$

From (I, 17) we see that if we take

$$\underset{p}{'v}{}^{q}_{b} = 0, \tag{22}$$

then the integrability conditions of (15) are identically satisfied and there exists an orthogonal transformation such that (22) is satisfied

for some set of normal vectors $\left({}'i_\lambda\right)$. For this of vectors, the corresponding ${}'h_{cb}$ become

$$
\begin{aligned}
(23) \qquad {}'h_{cb} &= {}'\alpha\, h_{cb}, \\
&= \left({}'\alpha\, h\right) i_c\, i_b, \\
&= {}'h\, i_c\, i_b.
\end{aligned}
$$

It is evident that ${}'h$ satisfies the Codazzi relation

$$
(24) \qquad \nabla'_{[c}\, {}'h_{b]a} = 0,
$$

and hence is equal in value to the $\left(h\right)$ of section IV, formula (38).

Since the fundamental quantities of the surface (V) coincide with those of $[1, 1]_a$, these former surfaces belong to the class $[1, 1]_a$. Thus we have the general theorem.

THEOREM V. *All the developable (applicable) V_3 in an R_6 which can be generated by ∞^1 flat R_2 are those of class $[1, 1]_a$. Conversely every V_3 of $[1, 1]_a$ is developable and can be generated by ∞^1 R_2 such that consecutive R_2 intersect. These V_3 are the envelopes of ∞^1 osculating R_3 to a V_1'' in an R_6 and conversely. This result is valid for V_3 in an R_n.*

VI. V_3 IN R_6 OF $[1, 1]_t$

The V_3 of this class satisfy the conditions

$$
(1) \qquad {}'v_\lambda = 0,
$$

$$
(2) \qquad h_{\mu\lambda} = h\, i_\mu\, i_\lambda,
$$

$$
(3) \qquad h_{\mu\lambda} = h\, i_\mu\, i_\lambda,
$$

$$
(4) \qquad h_{\mu\lambda} = h\, i_\mu\, i_\lambda,
$$

(5) i_λ is orthogonal to i_λ.

We must determine whether these V_3 in an R_6 are V_3 in an R_4 or an R_5. Assume that these V_3 lie in an R_4. Then by Theorem II, a set of $\binom{'i_\lambda}{p}$ exist for which

$$(6) \qquad \overset{q}{{}'v_\lambda} = 0, \qquad {}'h_{\underset{4}{\mu\lambda}} = {}'h_{\underset{5}{\mu\lambda}} = 0.$$

Thus by (I, 15)

$$(7) \qquad \nabla_\lambda \overset{q}{{}'\alpha} = 0.$$

The only solution of this system is

$$(8) \qquad \overset{q}{\alpha} = \text{constant}.$$

Upon substituting (6), (2), (3), (4) into (IV, 5) we obtain

$$(9) \qquad \overset{4}{\alpha}\, h\, i_\mu\, i_\lambda + \overset{5}{\alpha}\, h\, i_\mu\, i_\lambda + \overset{6}{\alpha}\, h\, i_\mu\, i_\lambda = 0,$$

$$(10) \qquad \overset{4}{\alpha}\, h\, i_\mu\, i_\lambda + \overset{5}{\alpha}\, h\, i_\mu\, i_\lambda + \overset{6}{\alpha}\, h\, i_\mu\, i_\lambda = 0.$$

From (5), these equations become

$$(11) \qquad \overset{4}{\alpha}\, h + \overset{5}{\alpha}\, h = \overset{6}{\alpha}\, h = 0,$$

$$(12) \qquad \overset{4}{\alpha}\, h + \overset{5}{\alpha}\, h = \overset{6}{\alpha}\, h = 0.$$

Conditions (8), (11), (12) must be satisfied by the $\overset{q}{h}$ in order that V_3 lie in an R_4. The only solutions of (11), (12) are

$$(13) \qquad \underset{6}{h} \neq 0, \qquad \underset{4}{h} = \underset{5}{h} = 0,$$

$$(14) \qquad \underset{6}{h} = 0, \qquad \underset{4}{h} \neq 0, \qquad \underset{5}{h} \neq 0.$$

Thus V_3 of $[1, 1]_5$ cannot lie in an R_4 under any condition. This is a unique feature as V_3 of $[1, 1]_6$ could be imbedded in an R_4 by proper selection of the principal curvatures $\binom{h}{p}$. By use of Theorem III, we can show that these V_3 lie in an R_5 only when

$$(15) \qquad \overset{q}{\underset{6}{\alpha}} = \text{constant}, \qquad (q = 4, 5, 6),$$

$$\underset{6}{\overset{4}{\alpha}}\underset{4}{h} + \underset{6}{\overset{5}{\alpha}}\underset{5}{h} = \underset{6}{\overset{6}{\alpha}}\underset{6}{h} = 0, \tag{16}$$

where of course, we must have:

$$\underset{4}{h}, \underset{5}{h}, \underset{6}{h} \neq 0. \tag{17}$$

We shall show in our study of the Codazzi relations that h can always be chosen so as to satisfy the Gauss, Codazzi, and Ricci relations yet not satisfy (15), (16).

We return to the general analysis of $[1, 1]_b$. Let (i^x) denote an arbitrary unit vector in the tangent R_3 at each point P. The curvature locus is given by

$$''u^x = \left(\underset{p}{h_{\mu\lambda}}\, i^\mu i^\lambda\right)\overset{p}{i^x}, \tag{18}$$

$$= \left(\underset{1}{i^\lambda}\, i_\lambda\right)^2\left(\underset{4}{h}\, i^x + \underset{5}{h}\, i^x\right) + \left(\underset{2}{i^\lambda}\, i_\lambda\right)^2\left(\underset{6}{h}\, i^x\right).$$

For asymptotic directions $\left(\underset{a}{i_\lambda}\right)$,

$$\underset{1\ a}{i^\lambda i_\lambda} = 0, \qquad \underset{2\ a}{i^\lambda i_\lambda} = 0. \tag{19}$$

For conjugate directions $\left(\underset{c}{i^x}, \underset{b}{i^x}\right)$,

$$''\underset{bc}{u^x} = \left(\underset{p}{h_{\mu\lambda}}\, \underset{c}{i^\mu}\, \underset{b}{i^\lambda}\right)\overset{p}{i^x} = 0, \tag{20}$$

$$\left(\underset{1\ c}{i^\lambda i_\lambda}\right)\left(\underset{1\ b}{i^\lambda i_\lambda}\right) = 0, \tag{21}$$

$$\left(\underset{2\ c}{i^\lambda i_\lambda}\right)\left(\underset{2\ b}{i^\lambda i_\lambda}\right) = 0. \tag{22}$$

Let us study (18), curvature locus. Let

$$\underset{4}{\bar{i}^x} = \frac{\underset{4\ 4}{h}\, i^x + \underset{5\ 5}{h}\, i^x}{\sqrt{\underset{4}{h^2} + \underset{5}{h^2}}}, \tag{23}$$

$$\underset{5}{\bar{i}^x} \text{ be orthogonal to } \underset{4}{\bar{i}^x}, \underset{6}{i^x}, \tag{25}$$

$$x^4 = \left(\underset{1}{i^\lambda}\, i_\lambda\right)^2\sqrt{\underset{4}{h^2} + \underset{5}{h^2}}, \tag{26}$$

$$x^6 = \left(\underset{2}{i^\lambda}\, i_\lambda\right)^2\underset{6}{h}, \tag{27}$$

then

$$''u^s = x^4 \underset{4}{i^s} + x^6 \underset{6}{i^s}. \tag{28}$$

If we select a direction $\underset{3}{i^s}$ in tangent R_3 such that

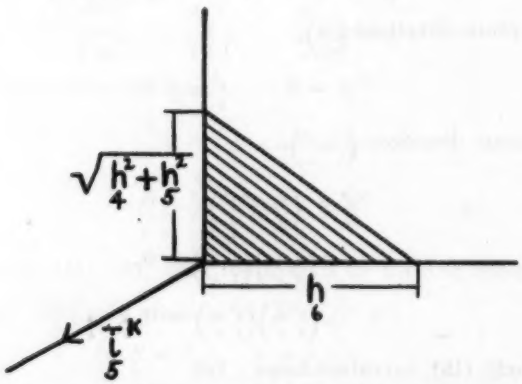
$$\underset{3}{i^s} \text{ is orthogonal } \underset{1}{i^s}, \underset{2}{i^s}, \tag{29}$$

then

$$\left(\underset{1}{i^\lambda}\, i_\lambda\right)^2 + \left(\underset{2}{i^\lambda}\, i_\lambda\right)^2 + \left(\underset{3}{i^\lambda}\, i_\lambda\right)^2 = 1, \tag{30}$$

$$\frac{x^4}{\sqrt{\underset{4}{h^2} + \underset{5}{h^2}}} + \frac{x^6}{\underset{6}{h}} + \left(\underset{3}{i^\lambda}\, i_\lambda\right)^2 = 1. \tag{31}$$

Thus from (26), (27) and (31) we see that the curvature locus appears as follows in the $\left(\underset{4}{i^\lambda}, \underset{6}{i^\lambda}\right)$ plane:



Furthermore, we see that to each direction in the tangent R_3 there corresponds a unique point in the curvature locus. To each point in the curvature locus, there corresponds $(2^3 = 8)$ directions in the tangent R_3. The direction $\underset{3}{i^s}$ which is orthogonal to $\underset{1}{i^s}, \underset{2}{i^s}$ is the *only asymptotic direction*. This direction $\underset{3}{i^s}$ is conjugate to every direction in the R_3. Any direction in the R_2 of $\left(\underset{2}{i^s}, \underset{3}{i^s}\right)$ is conjugate to any direction in the R_2 of $\left(\underset{1}{i^s}, \underset{3}{i^s}\right)$. These are the only conjugate directions.

We next analyze the Codazzi relations (I, 11, b). By use of equations similar to IV, (17–21), we find that: $\underset{1}{i}^*$, $\underset{2}{i}^*$, $\underset{3}{i}^*$ built triply orthogonal sets of V_2 (nets) in the V_3; the ∞^1 V_2 determined by $\left(\underset{2}{i}^*, \underset{3}{i}^*\right)$ are geodesic in the V_3; the ∞^1 V_2 determined by $\left(\underset{1}{i}^*, \underset{3}{i}^*\right)$ are geodesic in the V_3; hence, $\underset{3}{i}^*$ being asymptotic, must be a geodesic of an R_6 i.e. a straight line. If we adopt the net as a coördinate system, the linear element becomes

$$ds^2 = a'_{11}(du^1)^2 + a'_{22}(du^2)^2 + (du^3)^2. \tag{33}$$

And the conditions on the $\underset{p}{h}$ become

$$\frac{\partial \underset{p}{h}}{\partial u^c} = -\underset{p}{h}\frac{1}{2a'_{11}}\frac{\partial a'_{11}}{\partial u^c}, \qquad (p = 4, 5), (c = 2, 3), \tag{34}$$

$$\frac{\partial \underset{6}{h}}{\partial u^c} = -\underset{6}{h}\frac{1}{2a'_{22}}\frac{\partial a'_{22}}{\partial u^c}, \qquad (c = 1, 3). \tag{35}$$

Upon integration, we find

$$\underset{p}{h} = (a'_{11})^{-\frac{1}{2}} g_p(u^1), \qquad p = 4, 5, \tag{36}$$

$$\underset{6}{h} = (a'_{22})^{-\frac{1}{2}} f_6(u^2). \tag{37}$$

Where the functions g_p, f_6 are arbitrary. Since $\underset{p}{h}$ need only satisfy these conditions in order that the Gauss, Codazzi, and Ricci relations be satisfied, then conditions (15), (16) need not be satisfied. The theorem, covering this imbedment, is:

THEOREM VI. *If and only if the $g_p(u^1)$ of the above satisfy a linear relation with constant coefficients*

$$\sum_{6}^{4} \alpha g_4(u^1) + \sum_{6}^{5} \alpha g_5(u^1) = 0, \tag{38}$$

does a V_3 of $[1, 1]_6$ lie in an R_5.

We next study the Gauss relations (I, 11, a). Let

$$a_1 = \sqrt{a'_{11}}, \qquad a_2 = \sqrt{a'_{22}}. \tag{39}$$

Then by the formulae of Eisenhart,[5] the Gauss relations

$$K'_{abcd} = 0, \tag{40}$$

become

$$\frac{\partial^2 a_1}{\partial u^{3^2}} = 0, \tag{41}$$

$$\frac{\partial^2 a_2}{\partial u^{3^2}} = 0, \tag{42}$$

$$\frac{\partial}{\partial u^2}\left(\frac{1}{a_2}\frac{\partial a_1}{\partial u^2}\right) + \frac{\partial}{\partial u^1}\left(\frac{1}{a_1}\frac{\partial a_2}{\partial u^1}\right) + \frac{\partial a_1}{\partial u^3}\frac{\partial a_2}{\partial u^3} = 0, \tag{43}$$

$$\frac{\partial^2 a_2}{\partial u^1 \partial u^3} - \frac{1}{a_1}\frac{\partial a_2}{\partial u^1}\frac{\partial a_1}{\partial u^3} = 0, \tag{44}$$

$$\frac{\partial^2 a_1}{\partial u^2 \partial u^3} - \frac{1}{a_2}\frac{\partial a_1}{\partial u^2}\frac{\partial a_2}{\partial u^3} = 0. \tag{45}$$

From (41), (42), we have

$$a_1 = a + bu^3, \qquad a_2 = c + du^3, \tag{46}$$

where a, b, c, d are all functions of the variables (u^1, u^2). If we use the subscripts (1, 2) to denote differentiation with respect to u^1, u^2 in the symbols a, b, c, d, then (44) and (45) become

$$\underset{1}{d}(a + bu^3) = b\left(\underset{1}{c} + \underset{1}{d}u^3\right), \tag{47}$$

$$\underset{2}{b}(c + du^3) = d\left(\underset{2}{a} + \underset{2}{b}u^3\right). \tag{48}$$

By simplification, (47) and (48) become

$$\underset{1}{da} = c\underset{1}{b}, \tag{49}$$

$$\underset{2}{bc} = a\underset{2}{d}. \tag{50}$$

The set (43) becomes

$$\frac{\partial}{\partial u^2}\left(\frac{\underset{2}{a} + \underset{2}{b}u^3}{c + du^3}\right) + \frac{\partial}{\partial u^1}\left(\frac{\underset{1}{c} + \underset{1}{d}u^3}{a + bu^3}\right) + bd = 0. \tag{51}$$

If the equations (49), (50), (51) possess solutions for a, b, c, d then a linear element of *type* (33) exists which satisfies the Gauss relation (40). The following fact is very important. The fact that the linear element is of *type* (33) means that the Codazzi relations similar to IV, (17–20) are satisfied. Hence if a solution of (49), (50), (51) exists then V_3 of

$[1, 1]_b$ actually exist. We shall show that quantities (a, b, c, d) can be found so as to satisfy (49), (50), (51). In order to do this, we shall study the geometry of the V_3 of $[1, 1]_b$.

VII. Geometry of V_3 of $[1, 1]_b$

Since the V_2 built by $\binom{i^\lambda}{1}, \binom{i^\lambda}{3}$ and $\binom{i^\lambda}{2}, \binom{i^\lambda}{3}$ are geodesic in the V_3, they are developable. Furthermore since the V_3 is developable, we can bend the V_3 into an R_3. The straight line congruence $\binom{i^\lambda}{3}$ goes into a straight line congruence and hence the V_2 of the above are formed by the tangents to space curves in the V_3. Thus under bending, the V_2 of fields $\binom{i^\lambda}{1}, \binom{i^\lambda}{2}$ in the V_3 go into parallel V_2 in an R_3; the V_2 of $\binom{i^\lambda}{1}, \binom{i^\lambda}{3}$, $\binom{i^\lambda}{2}, \binom{i^\lambda}{3}$ become the principal surfaces of the rectilinear congruence. The linear element of such a system of surfaces is well known. Since this element is unaltered by bending, it goes into the linear element of the V_3. Thus we find this linear element, using the work of Weatherburn.[6] Let us fix one of the parallel V_2 as base surface. Let $\rho(u^1, u^2)$ denote the principal curvature of this surface along the (u^1) variable coördinate line. Let $\sigma(u^1, u^2)$ denote the principal curvature along the (u^2) coördinate line. Let E denote the a_{11} of this V_2; let G denote the a_{22} of this V_2. Then the linear element is:

$$(1) \qquad ds^2 = (\rho u^3 - 1)^2 E(du^1)^2 + (\sigma u^3 - 1)^2 G(du^2)^2 + (du^3)^2.$$

By comparing this with VI, (33), (39), (46), we see

$$(2) \qquad a + bu^3 = E^{\frac{1}{2}}\rho u^3 - E^{\frac{1}{2}},$$

$$(3) \qquad c + du^3 = G^{\frac{1}{2}}\sigma u^3 - G^{\frac{1}{2}}.$$

Thus

$$(4) \qquad a = -E^{\frac{1}{2}}, \qquad b = \rho E^{\frac{1}{2}},$$

$$(5) \qquad c = -G^{\frac{1}{2}}, \qquad d = \sigma G^{\frac{1}{2}}.$$

Then

$$(6) \qquad b = -\rho a\,, \qquad d = -\sigma c\,.$$

Let us eliminate (b, d) in VI, (49), (50), (51) by use of the variables (a, c, ρ, σ). The set (49), (50), (51) becomes

$$(7) \qquad \underset{1}{\sigma c} + (\sigma - \rho)\underset{1}{c} = 0,$$

$$\rho a_{2} + (\rho - \sigma) a_{2} = 0, \tag{8}$$

$$\frac{\partial}{\partial u^2}\left(\frac{a}{c}_{2}\right) + \frac{\partial}{\partial u^1}\left(\frac{c}{a}_{1}\right) + \sigma\rho ac = 0, \tag{9}$$

where we have assumed that no one of the quantities (a, c, ρ, σ) are zero.

Since a V_2 in an R_3 is determined when (a, c, ρ, σ) of (4), (5) satisfy the Gauss, Codazzi relations, we know that (a, c, ρ, σ) must satisfy these relations and (7), (8), (9). Except for these conditions (a, c, ρ, σ) may be arbitrary. We shall calculate these relations.

The fundamental quantities of the V_2 in an R_3 are given in terms of (a, c, ρ, σ) by the following equations:

$$\bar{a}_{11} = E = a^2, \qquad \bar{a}_{12} = 0, \qquad \bar{a}_{22} = G = c^2, \tag{10}$$

$$\bar{h}_{11} = L = \rho a^2, \qquad \bar{h}_{12} = 0, \qquad \bar{h}_{22} = N = \sigma c^2. \tag{11}$$

Substituting into the Gauss, Codazzi relations, these become

$$\sigma c_{1} + (\sigma - \rho) c_{1} = 0, \tag{12}$$

$$\rho a_{2} + (\rho - \sigma) a_{2} = 0, \tag{13}$$

$$\frac{\partial}{\partial u^2}\left(\frac{a}{c}_{2}\right) + \frac{\partial}{\partial u^1}\left(\frac{c}{a}_{1}\right) - \sigma\rho a^2 c^2 = 0. \tag{14}$$

Thus conditions (7), (8), (9) and the condition

$$a = -\frac{1}{c}, \tag{15}$$

form a complete system of conditions on the variables (a, c, ρ, σ), all of which are functions of (u^1, u^2). We can now easily show that (a, c, ρ, σ) can be found as functions of (u^1, u^2). In fact, take (ρ, σ) as arbitrary functions of (u^1, u^2), then (7) determines (c) to within a function of (u^2). Likewise (8) determines (a) to within a function of (u^1). Equation (15) gives a relation between these two unknown functions. Upon substituting in (9), we have a differential equation (partial) of the first order to solve for one of these functions. The other unknown function is then determined by (15). Then from (6), (b), (d) can be found. Thus, we have shown that quantities (a, b, c, d) satisfying VI (49), 50), (51) always exist. Hence the V_2 of $[1, 1]_4$ always exist.

VIII. V_3 in R_6 Generated by a Special V_2 in R_6 and a Normal Congruence

In this section, we follow the ideas of V, and consider geometric methods of generating V_3 of $[1, 1]_b$. Consider a V_2 in an R_6 whose linear element is given by

$$ds^2 = a^2(du^1)^2 + \frac{(du^2)^2}{(a^2)}, \tag{1}$$

and a congruence $(\overset{\alpha}{3})$ normal to V_2 and such that the consecutive congruence generators intersect along (u^1) and (u^2) coördinate lines. Then for the V_3 generated by this congruence,

$$y^\alpha = x^\alpha(u^1, u^2) + u^3 \overset{\alpha}{3}(u^1, u^3), \tag{2}$$

$$\frac{\partial y^\alpha}{\partial u^1} = \frac{\partial x^\alpha}{\partial u^1} + u^3 \frac{\partial \overset{\alpha}{3}}{\partial u^1}, \tag{3}$$

$$\frac{\partial y^\alpha}{\partial u^2} = \frac{\partial x^\alpha}{\partial u^2} + u^3 \frac{\partial \overset{\alpha}{3}}{\partial u^2}, \tag{4}$$

$$\frac{\partial y^\alpha}{\partial u^3} = \overset{\alpha}{3}. \tag{5}$$

Since

$$\sum_\lambda \frac{\partial x^\alpha}{\partial u^1}\frac{\partial x^\alpha}{\partial u^1} = a^2, \qquad \sum_\lambda \frac{\partial x^\alpha}{\partial u^1}\frac{\partial x^\alpha}{\partial u^2} = 0, \qquad \sum_\lambda \frac{\partial x^\alpha}{\partial u^2}\frac{\partial x^\alpha}{\partial u^2} = \frac{1}{a^2}, \tag{6}$$

$$\frac{\partial \overset{\alpha}{3}}{\partial u^1} = -\rho \frac{\partial x^\alpha}{\partial u^1}, \qquad \frac{\partial \overset{\alpha}{3}}{\partial u^2} = -\sigma \frac{\partial x^\alpha}{\partial u^2}. \tag{7}$$

Then the linear element of the V_3 becomes

$$a'_{11} = \sum_\alpha \frac{\partial y^\alpha}{\partial u^1}\frac{\partial y^\alpha}{\partial u^1} = a^2(1 - \rho u^3)^2, \tag{8}$$

$$a'_{22} = \sum_\alpha \frac{\partial y^\alpha}{\partial u^2}\frac{\partial y^\alpha}{\partial u^2} = \frac{1}{a^2}(1 - \sigma u^3)^2, \tag{9}$$

$$a'_{33} = \sum_\alpha \frac{\partial y^\alpha}{\partial u^3}\frac{\partial y^\alpha}{\partial u^3} = 1, \qquad a'_{bc} = 0, \qquad b \neq c. \tag{10}$$

Thus the linear element of these V_2 coincides with that of the V_2 of type $[1, 1]_b$.

We next study the second fundamental tensors of these V_2 in an R_6. First we study the second partial derivatives of (x^ϵ).

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{(\partial u^1)^2}\frac{\partial x^\epsilon}{\partial u^1} = a\,\underset{1}{a}, \tag{11}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{(\partial u^1)^2}\frac{\partial x^\epsilon}{\partial u^2} = -\sum_\epsilon \frac{\partial^2 x^\epsilon}{\partial u^1\,\partial u^2}\frac{\partial x^\epsilon}{\partial u^1}, \tag{12}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{(\partial u^1)^2}\underset{3}{i}^\epsilon = \rho a^2, \tag{13}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{(\partial u^2)^2}\frac{\partial x^\epsilon}{\partial u^1} = -\sum_\epsilon \frac{\partial^2 x^\epsilon}{\partial u^1\,\partial u^2}\frac{\partial x^\epsilon}{\partial u^2}, \tag{14}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{(\partial u^2)^2}\frac{\partial x^\epsilon}{\partial u^2} = c\,\underset{2}{c}, \qquad \left(c = \frac{1}{a}\right), \tag{15}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{(\partial u^2)^2}\underset{3}{i}^\epsilon = \sigma c^2, \tag{16}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{\partial u^1\,\partial u^2}\frac{\partial x^\epsilon}{\partial u^1} = a\,\underset{2}{a}, \tag{17}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{\partial u^1\,\partial u^2}\frac{\partial x^\epsilon}{\partial u^2} = c\,\underset{1}{c}, \tag{18}$$

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{\partial u^1\,\partial u^2}\underset{3}{i}^\epsilon = 0. \tag{19}$$

The Weingarten formulae for a V_2 in an R_6 can be written as follows (using a bar to denote V_2 quantities):

$$\frac{\partial^2 x^\epsilon}{\partial u^a\,\partial u^b} - \bar{\Gamma}^c_{ab}\frac{\partial x^\epsilon}{\partial u^c} = \underset{p}{\bar{h}}_{ab}\,\overset{p}{i}{}^\epsilon, \qquad (p = 3, 4, 5, 6). \tag{20}$$

Thus

$$\sum_\epsilon \frac{\partial^2 x^\epsilon}{\partial u^a\,\partial u^b}\overset{p}{i}{}^\epsilon = \underset{p}{\bar{h}}_{ab}, \qquad (a, b = 1, 2), (p = 3, 4, 5, 6). \tag{21}$$

We next list the second partials of (3), (4), (5) obtained by differentiation. Subscripts on a, c, ρ, σ denote differentiation.

$$\frac{\partial^2 y^s}{(\partial u^1)^2} = \frac{\partial^2 x^s}{(\partial u^1)^2}(1 - \rho u^3) - u^3 \rho_1 \frac{\partial x^s}{\partial u^1}, \tag{22}$$

$$\frac{\partial^2 y^s}{\partial u^1 \partial u^2} = \frac{\partial^2 x^s}{\partial u^1 \partial u^2}(1 - \rho u^3) - u^3 \rho_2 \frac{\partial x^s}{\partial u^1}, \tag{23}$$

$$\frac{\partial^2 y^s}{(\partial u^2)^2} = \frac{\partial^2 x^s}{(\partial u^2)^2}(1 - \sigma u^3) - u^3 \sigma_2 \frac{\partial x^s}{\partial u^2}, \tag{24}$$

$$\frac{\partial^2 y^s}{\partial u^2 \partial u^3} = -\sigma \frac{\partial x^s}{\partial u^2}, \tag{25}$$

$$\frac{\partial^2 y^s}{\partial u^1 \partial u^3} = -\rho \frac{\partial x^s}{\partial u^1}, \tag{26}$$

$$\frac{\partial^2 y^s}{(\partial u^3)^2} = 0, \tag{27}$$

$$\frac{\partial^2 y^s}{\partial u^2 \partial u^1} = \frac{\partial^2 x^s}{\partial u^2 \partial u^1}(1 - \sigma u^3) - u^3 \sigma_1 \frac{\partial x^s}{\partial u^2}. \tag{28}$$

As in (21), we find from the Weingarten formulae

$$\sum_s \frac{\partial^2 y^s}{\partial u^c \partial u^b} i^s_p = h_{cb}, \qquad (b, c = 1, 2, 3), (p = 4, 5, 6). \tag{29}$$

By studying (21), (29), and (11–19), we conclude

$$(30)\quad \overset{\bar{h}}{}_{11}^{3} = \rho a^2, \qquad \overset{\bar{h}}{}_{12}^{3} = 0, \qquad \overset{\bar{h}}{}_{22}^{3} = \sigma c^2,$$

$$(31)\quad h_{11}^{p} = \overset{\bar{h}}{}_{11}^{p}(1 - \rho u^3), \qquad h_{12}^{p} = 0, \qquad h_{22}^{p} = \overset{\bar{h}}{}_{22}^{p}(1 - \sigma u^3),$$

$$h_{a3}^{p} = 0, \qquad p = 4, 5, 6, \qquad a = 1, 2, 3.$$

If we require that

$$\overset{\bar{h}}{}_{22}^{p} = 0, \qquad p = 4, 5, \tag{32}$$

$$\overset{\bar{h}}{}_{11}^{p} = 0, \qquad p = 6, \tag{33}$$

then evidently the *second fundamental tensors of these V_3 coincide with those of* $[1, 1]_3$.

From the geometric properties of these V_3, we know that

$$\nabla'_{[c} \underset{p}{h}_{b]a} = 0, \tag{34}$$

is satisfied. Hence

$$\underset{p}{v}_{[c} \underset{q}{h}_{b]a} = 0. \tag{35}$$

By expanding these relations, using the results obtained for the $\left(\underset{q}{h}_{cb}\right)$, we find

$$\underset{q}{v}_{b} = 0. \tag{36}$$

Since

$$\begin{aligned} \underset{p}{v}_{b} &= B^{\epsilon}_{b} \left(\nabla_{\epsilon} \underset{p}{i}_{\lambda}\right) \underset{q}{i}^{\lambda}, \\ &= \underset{a}{i}_{b} \left(\underset{a}{i}^{\epsilon} \underset{q}{i}^{\lambda} \nabla_{\mu} \underset{p}{i}_{\lambda}\right). \end{aligned} \qquad (a = 1, 2, 3). \tag{37}$$

Thus from (36)

$$\underset{a}{i}^{\mu} \underset{q}{i}^{\lambda} \nabla_{\mu} \underset{p}{i}_{\lambda} = 0, \qquad a = 1, 2, 3, \quad q, p = 4, 5, 6. \tag{38}$$

Thus for the V_2 in an R_6

$$\underset{p}{\bar{v}}_{b} = 0, \qquad (q, p = 4, 5, 6). \tag{39}$$

From the equations

$$\bar{B}^{\epsilon}_{\lambda} = \underset{a}{i}^{\epsilon} \underset{a}{i}_{\lambda}, \qquad a = 1, 2, \tag{40}$$

$$\underset{p}{\bar{v}}_{b} = -\bar{B}^{\epsilon}_{b} \underset{p}{i}^{\lambda} \nabla_{\epsilon} \underset{3}{i}_{\lambda}, \tag{41}$$

$$\underset{1}{i}^{\mu} \nabla_{\mu} \underset{3}{i}^{\epsilon} = -\rho \underset{1}{i}^{\epsilon}, \qquad \underset{2}{i}^{\mu} \nabla_{\mu} \underset{3}{i}^{\epsilon} = -\sigma \underset{2}{i}^{\epsilon}, \tag{42}$$

we conclude

$$\underset{p}{\bar{v}}_{b} = 0. \tag{43}$$

THEOREM VII. The V_3 of $[1, 1]_5$ are generated by a V_2 in an R_4 with linear element

$$ds^2 = a^2(du^1)^2 + \frac{(du^2)^2}{a^2},$$

and a congruence $\left(\overset{i}{3}\right)$ normal to V_2 with consecutive congruence lines along (u^1, u^2) coördinate lines intersecting. Furthermore, we must require that the second fundamental tensors belonging to the remaining three normals be such that (32), (33) are satisfied. It can easily be *shown* [39, 43] that $\overset{q}{\underset{p}{v}}_\lambda = 0$ for this V_2.

IX. V_3 IN R_6 OF $[1, 1]_5$

The V_3 of this class satisfy the conditions

(1) $$\overset{q}{\underset{p}{v}}_\lambda = 0,$$

(2) $$\underset{4}{h}_{\lambda\alpha} = \underset{4\ 1\ 1}{h}\, i_\lambda\, i_\alpha,$$

(3) $$\underset{5}{h}_{\lambda\alpha} = \underset{5\ 2\ 2}{h}\, i_\lambda\, i_\alpha,$$

(4) $$\underset{6}{h}_{\lambda\alpha} = \underset{6\ 3\ 3}{h}\, i_\lambda\, i_\alpha,$$

(5) $\overset{i}{1}, \overset{i}{2}, \overset{i}{3}$ are mutually orthogonal.

We determine whether these V_3 lie in an R_4 or in an R_4, or an R_5. Assume that these V_3 lie in an R_4. Then by Theorem II, a set of $\left(\overset{'i}{p}\right)$ exist for which

(6) $$\overset{q}{\underset{p}{'v}}_\lambda = 0, \qquad \underset{4}{'h}_{\lambda\alpha} = \underset{5}{'h}_{\lambda\alpha} = 0.$$

Following the work of VI, 7–10, we conclude

(7) $$\overset{4}{\underset{4\ 4}{\alpha h}} = \overset{5}{\underset{4\ 5}{\alpha h}} = \overset{6}{\underset{4\ 6}{\alpha h}} = 0,$$

(8) $$\overset{4}{\underset{5\ 4}{\alpha h}} = \overset{5}{\underset{5\ 5}{\alpha h}} = \overset{6}{\underset{5\ 6}{\alpha h}} = 0.$$

Since the only solution of this system compatible with $\underset{q}{h}$ not zero is one such that Det. $\left|\underset{p}{\overset{q}{\alpha}}\right| = 0$, we see that the V_3 cannot lie in an R_4. Likewise by Theorem III, we see that the V_3 lies in an R_5 only when

$$\overset{q}{\underset{6}{\alpha}} = 0. \tag{9}$$

This is not possible, hence the V_3 cannot lie in an R_5. In fact the V_3 of [1, 1]$_e$ are the first type of V_3 that cannot be imbedded in an R_n $(n < 6)$.

We return to a study of the curvature vector,

$$''u^\alpha = \underset{4}{h}\left(\underset{1}{i^\lambda} i_\lambda\right)^2 \overset{4}{i}{}^\alpha + \underset{5}{h}\left(\underset{2}{i^\lambda} i_\lambda\right)^2 \overset{5}{i}{}^\alpha + \underset{6}{h}\left(\underset{3}{i^\lambda} i_\lambda\right)^2 \overset{6}{i}{}^\alpha. \tag{10}$$

For asymptotic directions $\left(\underset{a}{i_\lambda}\right)$,

$$\left(\underset{c}{i^\lambda} i_\lambda\right) = 0, \qquad c = 1, 2, 3. \tag{11}$$

For conjugate directions $\left(\underset{a}{i^\alpha}, \underset{b}{i^\alpha}\right)$,

$$\left(\underset{a}{i^\lambda} i_\lambda\right)\left(\underset{b}{i^\lambda} i_\lambda\right) = 0, \qquad c = 1, 2, 3. \tag{12}$$

The solutions of (12) are of the type

$$\underset{a}{i^\lambda} i_\lambda = \underset{a}{i^\lambda} i_\lambda = \underset{b}{i^\lambda} i_\lambda = 0. \tag{13}$$

Let us study (10) in greater detail. Place

$$x^4 = \underset{4}{h}\left(\underset{1}{i^\lambda} i_\lambda\right)^2, \qquad x^5 = \underset{5}{h}\left(\underset{2}{i^\lambda} i_\lambda\right)^2, \qquad x^6 = \underset{6}{h}\left(\underset{3}{i^\lambda} i_\lambda\right)^2. \tag{14}$$

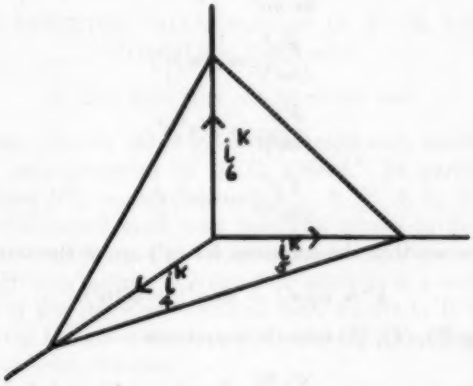
Then the curvature locus becomes

$$''u^\alpha = x^4 \overset{4}{i}{}^\alpha + x^5 \overset{5}{i}{}^\alpha + x^6 \overset{6}{i}{}^\alpha. \tag{15}$$

When referred to the $\left(\underset{4}{i^\alpha}, \underset{5}{i^\alpha}, \underset{6}{i^\alpha}\right)$ axes as a coördinate system in the normal space, the locus becomes

$$\frac{x^4}{\underset{4}{h}} + \frac{x^5}{\underset{5}{h}} + \frac{x^6}{\underset{6}{h}} = 1. \tag{16}$$

If we assume, $\left(\underset{4}{h}, \underset{5}{h}, \underset{6}{h} > 0\right)$ then geometrically this locus has following shape:



Evidently no asymptotic directions exist. The direction $\left(i^\alpha\right)$ is conjugate to any direction in the $\underset{2}{R}$ perpendicular to it ($\alpha = 1, 2, 3$). These are the only conjugate directions.

We next analyze the Codazzi relations (I, 11, 6). By use of equasions similar to IV, (17–21) we find: $\left(i^\alpha, i^\alpha, i^\alpha\right)$ built a triply orthogonal system of V_2 in the V_3; each such system of surfaces is geodesic in the V_3; hence the fields $\left(i^\alpha\right)$, which are the intersections of these V_2, are geodesics of the V_3. If we adopt the net as a coördinate system, the linear element becomes[7]

$$ds^2 = (du^1)^2 + (du^2)^2 + du^3)^2. \tag{18}$$

The conditions on $\underset{p}{h}$ become

$$\underset{4}{h} = f(u^1), \qquad \underset{5}{h} = g(u^2), \qquad \underset{6}{h} = k(u^3), \tag{19}$$

where all these functions are arbitrary. We note that there are no straight lines on this V_3. The fact that these V_3 exist follows from the fact that the (a'_{cb}) of (18) are a solution of the Gauss relation.

X. Integral Equations for the V_3 in an R_6 of [1, 1].

From the Weingarten formulae,

$$\frac{\partial^2 y^\epsilon}{\partial u^c \partial u^b} - \Gamma^a_{cb} \frac{\partial y^\epsilon}{\partial u^a} = \underset{p}{h}_{cb} i^\epsilon, \tag{1}$$

we find

$$\frac{\partial^2 y^\epsilon}{\partial u^a \partial u^b} = 0, \qquad a \neq b, \tag{2}$$

$$\frac{\partial^2 y^\epsilon}{(\partial u^1)^2} = f(u^1) i^\epsilon_4, \tag{3}$$

$$\frac{\partial^2 y^\epsilon}{(\partial u^2)^2} = g(u^2) i^\epsilon_5, \tag{4}$$

$$\frac{\partial^2 y^\epsilon}{(\partial u^3)^2} = k(u^3) i^\epsilon_6. \tag{5}$$

From (2), we see that the solutions for (y^ϵ) are of the form

$$y^\epsilon = \phi^\epsilon_1(u^1) + \phi^\epsilon_2(u^2) + \phi^\epsilon_3(u^3). \tag{6}$$

Substituting (3), (4), (5) into the equations

$$\sum_\epsilon \frac{\partial y^\epsilon}{\partial u^a} i^\epsilon_p = 0, \quad (q, p = 4, 5, 6, a = 1, 2, 3), \tag{7}$$

$$\sum_\epsilon i^\epsilon_p i^\epsilon_q = \delta_{pq}, \tag{8}$$

which fix the i^ϵ_p, we see that there are

$$(3 \times 3) + 6 = 15 \tag{9}$$

conditions on the first and second derivatives of the functions ϕ^ϵ_a. There are eighteen such functions.

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7. See (2), Page 58. This result is also evident by bending V_3 into R_3.

ON FOURIER TRANSFORMS IN L^p IN THE COMPLEX DOMAIN

By Loo-keng Hua and Shien-siu Shü[1]

Introduction. In this paper we are concerned with functions belonging to H^pL^p as introduced by A. C. Offord.[2] In particular, when $p = 2$, the class H^2L^2 is identical with L^2. R. E. A. C. Paley and N. Wiener[3] have obtained some very beautiful results in the case of L^2 on the Fourier transform of a function vanishing exponentially, of a function analytic in a strip, of a function analytic in a half-plane, etc. The purpose of this paper is to extend those results to H^pL^p. A difficulty in making the generalization is that for H^pL^p we have no corresponding Plancherel theorem

$$\int_{-\infty}^{\infty} |F(x)|^2\,dx = \int_{-\infty}^{\infty} |f(u)|^2\,du$$

where $F(x)$ is the Fourier transform of $f(u)$. The main results may be summarized as following:

1) If $f(u)$ is a function belonging to L^p in any finite interval and

$$f(x) = \begin{cases} O(e^{-\mu u}K(u)) & (u \to \infty), \\ O(e^{\lambda u}K(u)) & (u \to -\infty) \end{cases}$$

where $K(u)$ is a constant or a function of u belonging to $L^p(-\infty, \infty)$ then $f(u)$ belongs to H^pL^p, $p \geqq 2$ (Theorems 1 and 2).

2) If $F(s)$ is analytic in a strip, if $F(s)$ belongs uniformly to L^p in the strip and if $F(s)$ belongs to H^p on the boundaries of the strip, then for any interior point s, $F(s)$ can be expressed by the associated Cauchy integrals, and $F(s)$ belongs to H^pL^p in the strip (Theorems 3 and 5).

[1] National Tsing Hua University, Peiping, China.

[2] A. C. Offord, On Fourier Transforms III, Trans. Amer. Math. Soc. 38 (1935), pp. 250–266.

[3] Paley and Wiener, Fourier Transforms in the Complex Domain, 1934, Amer. Math. Soc. Colloquium Publication, vol. 19.

3) The conclusions of 2) are valid if we replace the condition that $F(s)$ belongs uniformly to L^p in the strip by the requirement that

$$F(s) = O\left(e^{e^{p|t|}}\right)$$

(Theorem of Phragmén-Lindelöf type; Theorem 7).

4) If $F(s)$ is analytic over the right half-plane if $F(s)$ belongs uniformly to L^p or

$$\lim_{r\to\infty} \frac{1}{r} \log |F(re^{\theta i})| = 0$$

uniformly over the right half-plane and if $F(it)$ belongs to H^p, then for any interior s, $F(s)$ can be expressed by an associated Cauchy integral and $F(s)$ belongs to H^pL^p. (Theorems 8 and 10.)

5) The two following classes of entire functions are identical:

(a) the class of entire functions $F(s)$ belonging to H^pL^p along the real axis and satisfying the condition

$$F(s) = O(e^{A|s|}),$$

(b) the class of all entire functions of the form $F(s) = \int_{-A}^{A} f(u)\, e^{ius}\, du$, where $f(u)$ belongs to L^p over $(-A, A)$ (Theorem 11).

6) If $F(z)$ is an entire function such that

$$\lim_{r\to\infty} \frac{1}{r} \overset{+}{\log} |F(re^{\theta i})| = 0$$

and does not vanish identically, it can not belong to H^pL^p along any line (Theorem 12).

We are indebted to Prof. N. Wiener for his lectures on Fourier Transforms at Tsing Hua University and we must take this opportunity to express our deep gratitude for his many valuable suggestions.

§1. In this section let us assume $f(u)$ to be a function belonging to L^p in any finite interval, $p \geqq 2$, and

$$f(u) = \begin{cases} O(e^{-\mu u} K(u)) & (u \to \infty), \\ O(e^{\lambda u} K(u)) & (u \to -\infty) \end{cases} \tag{1.1}$$

where $K(u)$ is a constant or a function belonging to $L^p(-\infty, \infty)$.

THEOREM 1. Let $\delta > 0$ and $-\lambda + \delta \leqq \sigma \leqq \mu - \delta$. Then

$$F(\sigma, t) = \int_{-\infty}^{\infty} f(u)\, e^{\sigma u - itu}\, du \tag{1.2}$$

exists and is bounded uniformly over the strip, and

$$F(\sigma, -t) = \overline{F(z)}, \qquad z = \sigma + it,$$

is an analytic function of z.

Proof. It is not very difficult to prove that $f(u)e^{\sigma u}$ belongs to L^{p_1} where $0 < p_1 \leqq p$. In particular, $p_1 = 1$, we see that (1.2) exists. Furthermore, over $-\lambda + \delta \leqq \sigma \leqq \mu - \delta$, we have

$$\begin{aligned} |F(\sigma, t)| &\leqq \int_{-\infty}^{\infty} |f(u)| e^{\sigma u} du \\ &\leqq \text{const.} \left[\int_{-\infty}^{-N} |K(u)| e^{\delta u} du + \int_{N}^{\infty} |K(u)| e^{-\delta u} du \right] \\ &\quad + \int_{-N}^{0} |f(u)| e^{-\lambda u} du + \int_{0}^{N} |f(u)| e^{\mu u} du \\ &= \text{const. (Independent of } \sigma \text{ and } t). \end{aligned}$$

Next let us consider the function

$$\begin{aligned} &\int_{-\infty}^{\infty} f(u) \left(\frac{e^{(z+\Delta z)u} - e^{zu}}{\Delta z} - ue^{zu} \right) du \\ &= \int_{-\infty}^{\infty} u f(u) e^{zu} \left(\frac{e^{\Delta z u} - 1}{\Delta z u} - 1 \right) du \\ &= \int_{-\infty}^{0} u f(u) e^{\left(z - \frac{\delta}{2}\right)u} e^{\frac{\delta}{2}u} \left(\frac{e^{\Delta z u} - 1}{\Delta z u} - 1 \right) du \\ &\quad + \int_{0}^{\infty} u f(u) e^{\left(z + \frac{\delta}{2}\right)u} e^{-\frac{\delta}{2}u} \left(\frac{e^{\Delta z u} - 1}{\Delta z u} - 1 \right) du. \end{aligned} \tag{1.3}$$

We are now going to prove that when $u < 0$, $|R(\Delta z)| < \frac{\delta}{2}$, we have

$$|G(u, \Delta z)| = \left| e^{\frac{\delta}{2}u} \left(\frac{e^{\Delta z u} - 1}{\Delta z u} - 1 \right) \right| \leqq 3.$$

In fact, when $|\Delta z u| \geqq 1$,

$$|G(u, \Delta z)| \leqq e^{R\left(\frac{\delta}{2} + \Delta z\right)u} + 2e^{\frac{\delta}{2}u} \leqq 3$$

when $|\Delta z u| < 1$

$$|G(u, \Delta z)| \leqq \left| \frac{u\Delta z}{2!} + \frac{(u\Delta z)^2}{3!} + \cdots \right|$$

$$\leqq \frac{1}{2!} + \frac{1}{3!} + \cdots \leqq e - 1 < 3.$$

Similarly when $u > 0$, $|R(\Delta z)| < \frac{\delta}{2}$, we have

$$\left| e^{-\frac{\delta}{2}u} \left(\frac{e^{\Delta z u} - 1}{\Delta z u} - 1 \right) \right| \leqq 3.$$

Using Lebesgue's convergence theorem we have that (1.3) tends to zero as $\Delta z \to 0$. Thus

$$\frac{d}{dz} \int_{-\infty}^{\infty} f(u)\, e^{zu}\, du = \int_{-\infty}^{\infty} f(u)\, u\, e^{zu}\, du.$$

THEOREM 2. Under the same hypotheses as before, we have

$$\int_{-\infty}^{\infty} |F(\sigma, t)|^p\, dt < C$$

where C is a constant independent of σ. Thus, by Offord's criterion of H^p, $f(u)e^{\sigma u}$ belongs to H^pL^p.[4] Consequently $F(\sigma, t)$ belongs to H^pL^p.[5]

Proof. Firstly, let us prove that there exists a number T such that $|t| > T$

$$|F(\sigma, t)| < 1$$

for any σ on $-\lambda + \delta' \leqq \sigma \leqq \mu - \delta$. Suppose this be false when $t \to \infty$. Let $t_1 < t_2 < \cdots < t_n \cdots$ be a sequence which tends to infinity. Then for each t_i, there is at least one $\sigma_i (-\lambda + \delta \leqq \delta_i \leqq \mu - \delta)$ and one $\tau_i (> t_i)$ such that

$$|F(\sigma_i, \tau_i)| \geqq 1.$$

Let σ_0 be one of the limiting points of $\{\sigma_i\}$. (It does exist by the Bolzano-Weiertrass theorem.)

[4] Loc. cit. Th. 9.
[5] Loc. cit. Th. 12.

By the Riemann-Lebesque Theorem there exists a number t_0, such that when $t > t_0$, we have

$$| F(\sigma_0, t) | < \tfrac{1}{4}.$$

On the other hand, since $F(\sigma, -t)$ is an analytic function of $\sigma + it$

$$| F(\sigma + \Delta\sigma, t) - F(\sigma, t) | \leqq \left| \int_\sigma^{\sigma+\Delta\sigma} F'_\sigma(\sigma, t)\, dt \right|$$

$$\leqq \int_\sigma^{\sigma+\Delta\sigma} \left| \int_{-\infty}^{\infty} uf(u)\, e^{(\sigma - it)u}\, du \right| dt.$$

The integrand is uniformly bounded over the strip $-\lambda + \delta \leqq \sigma \leqq \mu - \delta$. Hence we can choose δ_0, such that whenever

$$| \sigma - \sigma_0 | < \delta_0$$

we obtain

$$| F(\sigma, t) - F(\sigma_0, t) | < \tfrac{1}{4}$$

for any value of t. Therefore

$$| F(\sigma, t) | < \tfrac{1}{2} \qquad (t > t_0).$$

We can then choose m great enough, such that $| \sigma_m - \sigma_0 | < \delta_0$ and $t_m > t_0$, thus we have

$$F(\sigma_m, \tau_m) | < \tfrac{1}{2}.$$

This is a contradiction.

Now let us consider

$$\int_{-\infty}^{\infty} | F(\sigma, t) |^p\, dt = \left[\int_{-\infty}^{-\tau} + \int_{-\tau}^{\tau} + \int_{\tau}^{\infty} \right] | F(\sigma, t) |^p\, dt$$

$$\leqq \int_{-\tau}^{\tau} | F(\sigma, t) |^p\, dt + \left[\int_{-\infty}^{-\tau} + \int_{\tau}^{\infty} \right] | F(\sigma, t) |^2\, dt.$$

The first term is easily seen to be bounded, since its integrand is bounded by theorem 1. The second term is uniformly bounded over the given strip by the Plancherel theorem in L^2.

§2. THEOREM 3. If $F(s)$ is analytic over $-\lambda \leqq \sigma \leqq \mu$, $s = \sigma + it$ and

$$\int_{-\infty}^{\infty} | F(\sigma + it) |^p\, dt < \text{const.}$$

over this region, then when s is an interior point of this region, we have

$$(2.1)\quad F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{\mu+iy-s}\,dy - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(-\lambda+iy)}{-\lambda+iy-s}\,dy.$$

The proof of this theorem is quite similar to that given in N. Wiener and Paley's book (pp. 3–5) this could be done merely by using Hölder inequality instead of Schwarz inequality.

Consequently we have

THEOREM 4. Under the hypotheses of theorem 3, $F(s)$ is bounded over any region $-\lambda+\delta \leqq \sigma \leqq \mu-\delta$.

By the Hölder inequality we obtain that

$$\left|\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{\mu+iy-s}\,dy\right| \leqq \left\{\int_{-\infty}^{\infty}|F(\mu+iy)|^p\,dy\right\}^{\frac{1}{p}}\left\{\int_{-\infty}^{\infty}\frac{dy}{|\mu+iy-s|^q}\right\}^{\frac{1}{q}}$$

is bounded since $2 < p < \infty$ and $q = \dfrac{p}{p-1} > 1$. Similarly, the other term of (2.1) is also bounded.

THEOREM 5. Besides the hypotheses of theorem 3, we assume further that $F(\sigma+it)$ belongs to H^p for $\sigma = \mu$ and $-\lambda$, then there exists a measurable function $f(x)$ such that

$$(2.2)\quad \int_{-\infty}^{\infty}|f(x)|^p e^{p\mu x}\,dx < M; \qquad \int_{-\infty}^{\infty}|f(x)|^p e^{-p\lambda x}\,dx < M,$$

and that over the open interval $-\lambda < \sigma < \mu$, $F(\sigma+it)$ and $f(x)e^{\sigma x}$ are Fourier transforms of each other in L^p. Moreover, for each σ on this open interval

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,e^{(\sigma+it)u}\,du$$

converges in the ordinary sense to the function $F(\sigma+it)$ everywhere on $-\infty < t < \infty$.

Proof. Since $F(\sigma+it)$ belongs to H^pL^p for $\sigma = \mu$ and $\sigma = -\lambda$,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\mu+it)\,e^{-itx}\,dt$$

is summable $(C, 1)$ almost everywhere to a function $f(\mu, x)$ which is the Fourier transform of $F(\mu+it)$ in L^p and therefore it belongs also to H^pL^p. Similarly we define $f(-\lambda, t)$. (Offord, Theorem 12.)

Let us put

$$G(x) = \begin{cases} 0 & x < 0; \\ e^{-ax+itx} & x > 0;\ a > 0. \end{cases}$$

We obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x)\, e^{ixy}\, dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{ixy-(\alpha-it)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha - it - iy} = g(y)$$

$g(y)$ is bounded in any finite range. By theorems 1 and 2, $G(x)$ belongs to $H^q L^q$, $q = \dfrac{p}{p-1}$. By Offord's theorem 1,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-ixy}}{\alpha - i(t+y)}\, dy = \begin{cases} 0 & , x < 0, \\ e^{(-\alpha+it)x}, & x > 0. \end{cases} \qquad (c, 1)$$

Again by the use of Offord's theorem 3, when $-\lambda + \delta \leqq \sigma \leqq \mu - \delta$, and for any value of t we have

$$\int_{0}^{\infty} f(\mu, x)\, e^{(\sigma-\mu)x}\, e^{itx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{F(\mu+iy)}{\mu + iy - (\sigma + it)}\, dy$$

by putting $\alpha = \mu - \sigma$. It is easily seen that the two integrals of both sides converge in the ordinary sense. Similarly,

$$\int_{-\infty}^{0} f(-\lambda, x)\, e^{(\sigma+\lambda)x}\, e^{itx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{F(-\lambda+iy)}{-\lambda + iy - (\sigma + it)}\, dy.$$

By theorem 3, we get

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(-\lambda, x)\, e^{(\sigma+\lambda)x}\, e^{itx}\, dx$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(\mu, x)\, e^{(\sigma-\mu)x}\, e^{itx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{\sigma x}\, dx \qquad (2.3)$$

where

$$f(x) = \begin{cases} f(-\lambda, x)\, e^{\lambda x}, & (x < 0), \\ f(\mu, x)\, e^{-\mu x} & (x > 0) \end{cases} \qquad (2.4)$$

and (2.3) converges everywhere in t in the ordinary sense. By theorem 1, for $-\lambda < \sigma < \mu$, $f(x)\, e^{\sigma x}$ and $F(s)$ belong to $L^p H^p$, Moreover

$$\int_{-\infty}^{\infty} |f(x)|^p\, e^{p\sigma x}\, dx = \int_{-\infty}^{0} |f(-\lambda, x)|^p\, e^{p(\sigma+\lambda)x}\, dx + \int_{0}^{\infty} |f(\mu, x)|^p\, dx$$

$$\leqq \int_{-\infty}^{0} |f(-\lambda, x)|^p\, dx + \int_{0}^{\infty} |f(\mu, x)|^p\, dx$$

$$< \text{const.}$$

Similarly

$$\int_{-\infty}^{\infty} |f(x)|^p e^{-p\lambda x}\,dx < \text{const.}$$

§3. Theorems of the Phragmén-Lindelöf type.

THEOREM 6. If $F(\sigma + it)$ be analytic over $-\lambda \leqq \sigma \leqq \mu$, and $F(\sigma + it)$ belongs to H^pL^p for $\sigma = -\lambda$ and $\sigma = \mu$, $p \geqq 2$, and if

$$|F(\sigma + it)| < M$$

uniformly in the strip $-\lambda \leqq \sigma \leqq \mu$, then the conclusions of theorems 3 and 5 are valid.

Proof. The conclusion of theorem 3 is easily verified, just as shown in Paley and Wiener's book, p. 9.

In the proof of theorem 5, we have only used the facts that the conclusions of theorem 3 is valid and that $F(\sigma + it)$ belongs to H^pL^p for $\sigma = -\lambda$ and $\sigma = \mu$. These are true in the present case.

THEOREM 7. If $F(s)$ is an analytic function of s over the strip $-\lambda \leqq \sigma \leqq \mu$, if $F(s)$ belongs to H^pL^p $(p \geqq 2)$ for $\sigma = -\lambda$ and $\sigma = \mu$, and if

$$|F(\sigma + it)| = O\left(e^{e^{\rho|t|}}\right) \qquad (-\lambda \leqq \sigma \leqq \mu)$$

where $\rho < \pi/(\lambda + \mu)$, then the conclusions of theorems 3, 4 and 5 are valid.

Proof. Let us consider the function

$$F_\epsilon(\sigma + it) = \frac{1}{\epsilon}\int_t^{t+\epsilon} F(\sigma + it)\,dt$$

which satisfies the following four properties:

(1) $$F_\epsilon(\sigma + it) = O\left(e^{e^{\rho'|t|}}\right) \qquad (-\lambda < \sigma < \mu),$$

(2) $$|F_\epsilon(\mu + it)| \leqq \frac{1}{\epsilon^{1-\frac{1}{q}}}\left[\int_t^{t+\epsilon} |F(\mu + it)|^p\,dt\right]^{\frac{1}{p}},$$

$$|F_\epsilon(-\lambda + it)| \leqq \frac{1}{\epsilon^{1-\frac{1}{q}}}\left[\int_t^{t+\epsilon} |F(-\lambda + it)|^p\,dt\right]^{\frac{1}{p}}, \qquad q = \frac{p-1}{p}.$$

Thus, for each $\epsilon > 0$, $F_\epsilon(\sigma + it)$ is uniformly bounded over $-\lambda \leqq \sigma \leqq \mu$ by the classical Phragmén-Lindelöf theorem.

(3) $F_\epsilon(\mu + it)$ and $F_\epsilon(-\lambda + it)$ belong to L^p uniformly in ϵ. Since by the Hölder inequality, we obtain

$$\int_a^b \left| \frac{1}{\epsilon} \int_t^{t+\epsilon} F(\mu + iy)\, dy \right|^p dt \leqq \frac{1}{\epsilon^{\frac{p}{q}}} \int_a^b dt \int_t^{t+\epsilon} |F(\mu + iy)|^p\, dy$$

$$= \frac{1}{\epsilon} \int_{a+\epsilon}^b dy \int_{y-\epsilon}^y |F(\mu + iy)|^p\, dt + \frac{1}{\epsilon} \int_a^{a+\epsilon} dy \int_a^y |F(\mu + iy)|^p\, dt$$

$$+ \frac{1}{\epsilon} \int_b^{b+\epsilon} dy \int_{y-\epsilon}^b |F(\mu + iy)|^p\, dt < \int_a^{b+\epsilon} |F(\mu + iy)|^p\, dy$$

$$\leqq \int_{-\infty}^{\infty} |F(\mu + iy)|^p\, dy.$$

Similarly, $F_\epsilon(-\lambda + it)$ belongs to L^p uniformly in ϵ.

(4) For each $\epsilon > 0$, $F_\epsilon(\mu + it)$ and $F_\epsilon(-\lambda + it)$ belong to H^p. Let us put

$$F(\mu + it) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mu, x)\, e^{ixt}\, dx \qquad (C, 1)$$

where $f(\mu, x)$ belongs to H^pL^p. Therefore

$$F(\mu + it) - \frac{1}{\sqrt{2\pi}} \int_{-w}^{w} \left(1 - \frac{|x|}{w}\right) f(\mu, x)\, e^{itx}\, dx = o(1)$$

as $w \to \infty$. Thus we have

$$F_\epsilon(\mu + it) - \frac{1}{\sqrt{2\pi}} \int_{-w}^{w} \left(1 - \frac{|x|}{w}\right) f(\mu, x)\, \frac{e^{ix\epsilon} - 1}{ix\epsilon}\, e^{ixt}\, dx = o(1)$$

almost everywhere in t. Hence

$$F_\epsilon(\mu + it) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mu, x)\, \frac{e^{ix\epsilon} - 1}{ix\epsilon}\, e^{ixt}\, dx.$$

The right integral converges everywhere to a function which is finite everywhere and belongs to L in every finite range, then by Offord's theorems 8 and 5, we obtain that the function

$$\frac{f(\mu, x)}{ix} (e^{ix\epsilon} - 1)$$

belongs to L_p^* and $F_\epsilon(\mu + it)$ belongs to H^p.

Then by the use of the previous theorem we have

$$F_\epsilon(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(\mu+iy)}{\mu+iy-s}\,dy - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(-\lambda+iy)}{-\lambda+iy-s}\,dy, \quad -\lambda < \sigma < \mu.$$

On account of the fact that both of the integrals converges uniformly in ϵ, let ϵ tend to zero we have

$$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(\mu+iy)}{\mu+iy-s}\,dy - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(-\lambda+iy)}{-\lambda+iy-s}\,dy.$$

Thus, together with the conditions that $F(\mu+it)$ and $F(-\lambda+it)$ belong to H^pL^p, we obtain the conclusions of theorems 4 and 5.

§4. The Fourier transform of a function in a half-plane.

THEOREM 8. If $F(\sigma+it)$ is analytic for $\sigma \geqq 0$ and $F(it)$ belongs to H^p, and if

$$\int_{-\infty}^{\infty} |F(\sigma+it)|^p\,dt < \text{const.}$$

for $0 \leqq \sigma < \infty$, then the formula

$$F(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(iy)}{iy-s}$$

holds for $R(s) > 0$. Moreover, there exists a function $f(x)$ which vanishes almost everywhere for $x > 0$ and which belongs to $L^p(-\infty, 0)$ such that for $\sigma > 0$

$$F(\sigma+it) = \int_{-\infty}^{0} f(x)\,e^{x(\sigma+it)}\,dx$$

where the integral converges everywhere in t in the ordinary sense. $F(\sigma+it)$ and $f(x)e^{\sigma x}$ are Fourier transforms of each other in L^p for each $\sigma > 0$, and both of them belong to H^pL^p.

Proof. By theorem 3, we establish the formula

$$F(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(iy)}{iy-s}\,dy.$$

since

$$\left|\int_{-\infty}^{\infty} \frac{F(\mu+iy)}{\mu+iy-s}\,dy\right| \leqq \text{const.}\left\{\int_{-\infty}^{\infty} \frac{dy}{|\mu+iy-s|^q}\right\}^{\frac{1}{q}} = o(1)$$

as $\mu \to \infty$.

Now, just as we have done on theorem 5, we can prove that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(0, x)\, e^{sx+itx}\, dx$$

where $f(0, x)$ is the Fourier transform of $F(it)$ in L^p.

We define

$$f(x) = \begin{cases} f(0, x) & (x < 0), \\ 0 & (x > 0). \end{cases}$$

Therefore

$$\int_{-\infty}^{\infty} |f(x)|^p\, dx = \int_{-\infty}^{0} |f(0, x)|^p\, dx < \infty.$$

And by theorems 1 and 2, $f(x)\, e^{sx}$ belongs to H^pL^p. So is $F(\sigma + it)$ for each $\sigma > 0$.

THEOREM 9. If $F(s)$ is analytic over $0 \leqq \sigma < \infty$, if

$$|F(s)| < M$$

uniformly over $\sigma \geqq 0$, and if $F(it)$ belongs to H^pL^p, then the conclusions of theorem 8 are valid.

Proof. Just as we have done before, for $\sigma > 0$ we have

$$F(s) = \frac{1}{2\pi} \lim_{B \to \infty} \int_{-B}^{B} \frac{F(\mu + iy)}{\mu + iy - s}\, dy - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)}{iy - s}\, dy \tag{4.1}$$

where μ is any number greater than σ. Let us consider

$$\frac{1}{2\pi i} \int_{B}^{B+1} dA \left[\int_{-Ai}^{\mu - Ai} + \int_{\mu - Ai}^{\mu + Ai} + \int_{\mu + Ai}^{Ai} + \int_{Ai}^{i(1+\rho)} + \int_{i(1-\rho)}^{-Ai} \right] \frac{F(z)}{z - it}\, dz + \frac{1}{2\pi i} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\rho e^{\theta i} + it)\, id\theta = 0 \tag{4.2}$$

where $\rho e^{\theta i} = z - it$. In the first place,

$$\frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\rho e^{\theta i} + it)\, d\theta = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(it)\, dt$$

$$+ \frac{\rho}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{F(\rho e^{i\theta} + it) - F(it)}{\rho e^{i\theta}}\, e^{\theta i}\, d\theta = \frac{F(it)}{2} + O(\rho)$$

as ρ tends to zero since $F(z)$ is analytic at $z = it$.

As before, let B tend to infinity then (4.2) becomes

$$\frac{1}{2\pi}\left\{\lim_{B\to\infty}\int_{-B}^{B}\frac{F(\mu+iy)}{\mu+iy-it}\,dy-\left[\int_{-B}^{t-\rho}+\int_{t+\rho}^{B}\right]\frac{F(iy)}{i(y-t)}\,dy\right\}+\tfrac{1}{2}F(iy)+O(\rho)=0. \tag{4.3}$$

Since $F(iy)$ belongs to L^p, by a theorem due to Riesz,[6] when ρ tends to zero, the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(iy)}{y-t}\,dy$$

converges almost everywhere in t to a function $g(t)$ which belongs to L^p (in the sense of the principal value of Cauchy). Again,

$$\lim_{B\to\infty}\int_{-B}^{B}\left[\frac{F(\mu+iy)}{\mu+iy-s}-\frac{F(\mu+it)}{\mu+iy-it}\right]dy \leqq \text{const.}\int_{-\infty}^{\infty}\frac{dy}{|\mu+iy-s||\mu+iy-it|}=o(1) \tag{4.4}$$

as μ tends to infinity.

Together with (4.1), (4.3) and (4.4), we obtain

$$F(\sigma+it)=\text{const.}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(iy)\,dy}{iy-\sigma-it}=g(t)-\tfrac{1}{2}F(it)-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(iy)}{iy-\sigma-it}\,dy. \tag{4.5}$$

Because $g(t)-F(it)$ belongs to L^p, the constant in (4.5) must vanish and we have the formula

$$F(\sigma+it)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(iy)}{iy-\sigma-it}\,dy.$$

The other parts of the conclusion follow immediately as we have indicated in the preceding theorem.

By an argument similar to the one we have used in theorem 7, we obtain the following theorem of Phragmén-Lindelöf type:

THEOREM 10. If $F(s)$ is analytic over $0\leqq\sigma<\infty$, if

$$\lim_{r\to\infty}\frac{1}{r}\log|F(re^{\theta i})|=0$$

[6] M. Riesz, Sur les fonctions conjuguées, Math. Zeits. 27 (1928), pp. 218–244.

and if $F(it)$ belongs to H^pL^p, then the conclusions of theorem 8 are valid.

§5. Entire functions of exponential type.

THEOREM 11. The two following classes of entire functions are identical:

(1) the class of entire functions $F(z)$ belonging to H^pL^p along real axis and satisfying the condition

$$F(z) = O(e^{A|z|});$$

(2) the class of all entire functions of the form

$$F(z) = \int_{-A}^{A} f(u)\, e^{iuz}\, du$$

where $f(u)$ belongs to L^p over $(-A, A)$.

Proof. The class (2) is contained in the class (1). In the first place, $F(x)$ belongs to H^pL^p by theorems 1 and 2. In the second place

$$\begin{aligned} |F(z)| &\leqq \left\{\int_{-A}^{A} |f(u)|^p\, du\right\}^{\frac{1}{p}} \left\{\int_{-A}^{A} |e^{iqus}|\, du\right\}^{\frac{1}{q}} \\ &= \text{const.} \left\{\int_{-A}^{A} e^{qu|Iz|}\, du\right\}^{\frac{1}{q}} \\ &= \text{const.} \left\{\frac{e^{qA|Iz|} - e^{-qA|Iz|}}{q|Iz|}\right\}^{\frac{1}{q}} \\ &= O(e^{A|z|}). \end{aligned}$$

On the other hand, the class of (1) is of the form (2). In order to prove this, let us consider the function

$$G(z) = \frac{e^{-Az}}{\epsilon} \int_{z}^{z+i\epsilon} F(iw)\, dw$$

which is bounded over the imaginary axis and positive real axis, and which is at most of exponential growth. By the Phragmén-Lindelöf theorem, it is bounded on the right half-plane. As we have done for theorem 7, $G(it)$ belongs to H^pL^p. Hence by theorem 8 there exists $f_\epsilon(x)$ belonging to $L^p(-\infty, 0)$ such that

$$G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f_\epsilon(x)\, e^{zx}\, dx \qquad R(z) > 0 \tag{5.1}$$

and $G(z)$ belongs to H^pL^p for each $x = R(z) > 0$. Thus we have

$$F_\epsilon(z) = \frac{1}{\epsilon}\int_z^{z+\epsilon} F(w)dw = iG(iz)\,e^{iAz}$$
$$= \int_{-\infty}^{A} if_\epsilon(x - A)\,e^{izx} \qquad (I(z) < 0).$$

Similarly, we obtain

$$\frac{1}{\epsilon}\int_z^{z+\epsilon} F(w)dw = \int_{-A}^{\infty} i\bar{f}_\epsilon(x + A)\,e^{izx}\,dx \tag{5.2}$$

for $I(z) > 0$.

Let $H(z) = F_\epsilon(-iz)$. Then

$$H(z) = \int_{-A}^{\infty} i\bar{f}_\epsilon(x + A)\,e^{zx}\,dx \qquad (R(z) > 0),$$
$$= \int_{-\infty}^{A} if_\epsilon(x - A)\,e^{zx}\,dx \qquad (R(z) < 0). \tag{5.3}$$

Since $F(z)$ is at most of exponential growth, so is $H(z)$; because of (5.3), $H(z)$ belongs to H^pL^p for each $\sigma = Rz \neq 0$. Thus by theorem 7 $H(it)$ belongs to H^pL^p in t and there exists a function $g_\epsilon(x)$ such that

$$\frac{1}{\epsilon}\int_z^{z+\epsilon} F(w)\,dw = \int_{-\infty}^{\infty} g_\epsilon(u)\,e^{izu}\,du \qquad (C, 1) \tag{5.4}$$

for any z. Comparing with (2.4) and (5.3) we have

$$g_\epsilon(x) = \begin{cases} i\bar{f}_\epsilon(x + A) & \text{for } -A \leqq x < 0, \\ if_\epsilon(x - A) & \text{for } A \geqq x > 0, \\ 0 & \text{for } |x| > A. \end{cases}$$

In particular,

$$\frac{1}{\epsilon}\int_x^{x+\epsilon} F(w)\,dw = \int_{-A}^{A} g_\epsilon(u)\,e^{ixu}\,du.$$

On the other hand since $F(x)$ belongs to H^pL^p, we have

$$F(x) = \int_{-\infty}^{\infty} f(u)\,e^{ixu}\,du, \qquad (C, 1)$$

and

$$\frac{1}{\epsilon}\int_x^{x+\epsilon} F(x)\,dx = \int_{-\infty}^{\infty} f(u)\,\frac{e^{i\epsilon u} - 1}{i\epsilon u}\,e^{ixu}\,du.$$

Hence comparing with (5.4) $f(u)$ must vanish outside $(-A, A)$. Therefore, as ϵ tends to zero, (5.4) becomes

$$F(z) = \int_{-A}^{A} f(u)\, e^{izu}\, du.$$

An immediate corollary is

THEOREM 12. If $F(z)$ is an entire function such that

$$\lim_{r \to \infty} \frac{1}{r} \log^{+} | F(re^{i\theta}) | = 0$$

and does not vanish identically, it can not belong to H^pL^p along any line.

If it does, we may take this line to be the real axis. Thus for every $A > 0$, the Fourier transform of $F(x)$ will vanish almost everywhere outside $(-A, A)$ by the preceding theorem. Hence $F(z)$ must vanish identically and this contradicts our hypothesis.

ON THE NORMALIZATION OF THE SET OF FUNCTIONS $x^n \operatorname{sech} x$

By C. T. Chuang[1]

1. We propose to normalize the set of functions,

$$x^n \operatorname{sech} x, \qquad n = 0, 1, 2, \cdots, \infty,$$

by making use of their Fourier transforms $g_n(u)$ which are easily proved to be the following:

$$g_n(u) = (-1)^n (i)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{du^n} \operatorname{sech} \frac{\pi u}{2}, \quad n = 0, 1, 2, \cdots, \infty.$$

In fact, for $n = 0$ it is known[2] that

$$\sqrt{\frac{\pi}{2}} \operatorname{sech} \frac{\pi u}{2} = \operatorname*{l.i.m}_{A \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \operatorname{sech} x \, e^{iux} \, dx,$$

and by differentiating this n times we get $g_n(u)$. In carrying out the differentiation, we find that $g_n(u)$ is of the form, $G_n\left(\tanh \frac{\pi u}{2}\right) \operatorname{sech} \frac{\pi u}{2}$, where $G_n(y)$ is a polynomial in y of degree n and is an even or odd function according to that n is even or odd. We can then express the function $\left(\tanh \frac{\pi u}{2}\right)^n \operatorname{sech} \frac{\pi u}{2}$ as a linear combination of

$$g_0(u), g_2(u), \cdots, g_{2m}(u)$$

or of $g_1(u), g_3(u), \cdots, g_{2m+1}(u)$ according as n is equal to $2m$ or $2m + 1$. Let $P_n(y)$ be Legendre's polynomial of degree n. We see immediately that the function $P_n\left(\tanh \frac{\pi u}{2}\right) \operatorname{sech} \frac{\pi u}{2}$ can be expressed as a linear combination of the functions $g_k(u)$ in the same manner. On the other hand, from the well known relations,

$$\int_{-1}^{1} P_n(y) P_m(y) \, dy = \begin{cases} 0, & (n \neq m) \\ \dfrac{2}{2n+1}, & (n = m), \end{cases}$$

[1] National Tsing Hua University, Peiping, China.

[2] Raymond E. A. C. Paley and N. Wiener, Fourier transforms in the complex domain, p. 41.

we have on changing y into $\tanh \frac{\pi u}{2}$,

$$\frac{\pi}{2}\int_{-\infty}^{\infty} P_n\left(\tanh\frac{\pi u}{2}\right) P_m\left(\tanh\frac{\pi u}{2}\right)\operatorname{sech}^2\frac{\pi u}{2}\,du = \begin{cases} 0, & (n \neq m) \\ \dfrac{2}{2n+1}, & (n = m). \end{cases}$$

Hence the normalization of the set of functions $g_n(u)$, $n = 0, 1, 2, \cdots, \infty$ leads to the functions,

$$\frac{\sqrt{\pi(2n+1)}}{2} P_n\left(\tanh\frac{\pi u}{2}\right)\operatorname{sech}\frac{\pi u}{2}, \quad n = 0, 1, 2, \cdots, \infty.$$

2. By Plancherel's theorem[3] we have then

$$Q_n(x)\operatorname{sech} x = \operatorname*{l.i.m}_{A\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} \frac{\sqrt{\pi(2n+1)}}{2} P_n\left(\tanh\frac{\pi u}{2}\right)\operatorname{sech}\frac{\pi u}{2}\, e^{-iux}\,du, \tag{1}$$

where $Q_n(x)$ is a polynomial in x of degree n and is an even or odd function according to that n is even or odd. As we know, the set of functions $Q_n(x)\operatorname{sech} x$ is also normal and orthogonal over $(-\infty, \infty)$. In order to complete the normalization of the set of functions $x^n \operatorname{sech} x$, it remains then to find out the explicit form of $Q_n(x)$. We have

$$\operatorname*{l.i.m}_{A\to\infty}\int_{-A}^{A} P_n\left(\tanh\frac{\pi u}{2}\right)\operatorname{sech}\frac{\pi u}{2}\, e^{-iux}\,du = \int_{-\infty}^{\infty} P_n\left(\tanh\frac{\pi u}{2}\right)\operatorname{sech}\frac{\pi u}{2}\, e^{-iux}\,du. \tag{2}$$

On changing $e^{\pi u}$ into t, the integral on the right of the above relation becomes

$$\frac{2}{\pi}\int_0^{\infty} P_n\left(\frac{t-1}{t+1}\right)\frac{t^{-\frac{1}{2}-\frac{ix}{\pi}}}{t+1}\,dt. \tag{3}$$

But we have

$$P_n\left(\frac{t-1}{t+1}\right) = P_n\left(1-\frac{2}{t+1}\right) = \sum_{k=0}^{n}(-1)^k\frac{2^k}{k!}\frac{P_n^{(k)}(1)}{(t+1)^k},$$

[3].Loc. cit. p. 2.

where the values $P_n^{(k)}(1)$, $k = 0, 1, \cdots, n$, are found to be

$$P_n^{(k)}(1) = \frac{1}{2^k\, k!}\frac{(n+k)!}{(n-k)!}.$$

To evaluate the integral (3), it is then sufficient to compute the value of the integral,

$$\int_0^\infty \frac{t^{-\frac{1}{2}-\frac{ix}{\pi}}}{(t+1)^{k+1}}\,dt, \tag{4}$$

for $k = 0, 1, \cdots, n$. On changing $(1+t)^{-1}$ into ξ, we see that this integral is equal to

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where $p = k + \frac{1}{2} + \frac{ix}{\pi}$ and $q = \frac{1}{2} - \frac{ix}{\pi}$, from which the value of integral (4) is found to be

$$\frac{\pi}{2^k\, k!}\left(1+\frac{2ix}{\pi}\right)\left(3+\frac{2ix}{\pi}\right)\cdots\left(2k-1+\frac{2ix}{\pi}\right)\operatorname{sech} x.$$

The integral (3) is then easily evaluated. From its value and by (2) and (1), we get finally the result:

The set of functions $Q_n(x)\operatorname{sech} x$, $n = 0, 1, 2, \cdots, \infty$, where

$$Q_n(x) = \sqrt{n+\tfrac{1}{2}}\sum_{k=0}^{n}\frac{(-1)^k}{2^k(k!)^2}\frac{(n+k)!}{(n-k)!}\left(1+\frac{2ix}{\pi}\right)\left(3+\frac{2ix}{\pi}\right)\cdots\left(2k-1+\frac{2ix}{\pi}\right),$$

is normal and orthogonal over $(-\infty, \infty)$.

3. The function $\varphi(x) = Q_n(x)\operatorname{sech} x$ satisfies a certain integral equation which can be obtained in the following manner: We have the relation,

$$\frac{\sqrt{\pi(2n+1)}}{2}P_n\left(\tanh\frac{\pi u}{2}\right)\operatorname{sech}\frac{\pi u}{2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Q_n(x)\operatorname{sech} x\, e^{iux}\,dx,$$

or in setting $f(u) = P_n\left(\tanh\frac{\pi u}{2}\right)$,

$$f(u) = c\int_{-\infty}^{\infty}\varphi(x)\cosh\frac{\pi u}{2}\,e^{iux}\,dx,$$

where c is a constant. By the well known differential equation of $P_n(y)$, it is easily verified that $f(u)$ satisfies the equation,

$$\frac{d^2f}{du^2} + \frac{\pi^2}{4} n(n+1) \operatorname{sech}^2 \frac{\pi u}{2} f = 0. \tag{5}$$

But we have

$$\frac{\partial^2}{\partial u^2} e^{iux} \cosh \frac{\pi u}{2} + \frac{\pi^2}{4} n(n+1) \operatorname{sech}^2 \frac{\pi u}{2} e^{iux} \cosh \frac{\pi u}{2}$$

$$= \frac{\pi^2}{4} e^{iux} \cosh \frac{\pi u}{2} \left\{ \left(\frac{2ix}{\pi} + \tanh \frac{\pi u}{2} \right)^2 + (n^2 + n + 1) \operatorname{sech}^2 \frac{\pi u}{2} \right\},$$

from this and (5), it follows that $\varphi(x) = Q_n(x) \operatorname{sech} x$ satisfies the integral equation,

$$\int_{-\infty}^{\infty} \varphi(x) K_n(x, u) e^{iux} dx = 0,$$

where $K_n(x, u) = \left(\frac{2ix}{\pi} + \tanh \frac{\pi u}{2} \right)^2 + (n^2 + n + 1) \operatorname{sech}^2 \frac{\pi u}{2}$.

The writer wishes to thank Prof. N. Wiener for suggesting the problem and his kind guidance during the course of the work.

THREE-DIMENSIONAL CONTACT TRANSFORMATIONS IN SOLVED FORM

By Charles K. Robbins[1]

INTRODUCTION

The most general contact transformation in three dimensions[2] is defined by

$$x' = A(x, y, z, p, q)$$
$$y' = B(x, y, z, p, q)$$
$$z' = C(x, y, z, p, q)$$
$$p' = P(x, y, z, p, q)$$
$$q' = Q(x, y, z, p, q)$$

with the condition that $p'\,dx' + q'\,dy' - dz' = 0$ is a consequence of $p\,dx + q\,dy - dz = 0$. In geometrical language this transformation carries a union of plane elements into such a union, a plane element being defined by a point (x, y, z) and a direction $p, q, -1$. Line elements need not be considered because there are, in general, no contact transformations which carry a union of line elements into such a union. This last statement excepts, of course, the extended point transformations which will not be considered in this investigation. (In other words not all of the functions Ap, Aq, Bp, Bq, Cp, Cq shall vanish identically.)

The object of this study is to obtain some general types of contact transformations in the above solved form. Although this subject has an extended literature,[3] it seems that the problem has not been attacked

[1] Purdue University.

[2] The subject of two dimensional contact transformations in solved form has been treated by the author in a paper published in the American Mathematical Monthly, Volume XLIII, May, 1936.

[3] The work of Sophus Lie and his pupils. Also the original work of Jacobi in his Vorlesungen Über Dynamik and Articles by Liebmann and Prange in Encyklopädie der Mathematischen Wissenschaften.

systematically from the view point of explicit[4] representation. The results obtained are set forth as six general types of contact transformations in each of which the functions A, B, C, P and Q are explicitly obtainable. Only reversible transformations are considered, that is those in which the Jacobian (J) of the functions involved does not vanish identically.

As a first step in attacking the problem, let us apply explicitly the condition that $p'\,dx' + q'\,dy' - dz' = 0$ is a consequence of

$$p\,dx + q\,dy - dz = 0.$$

Then we have

$$\begin{aligned}(PA_x + QB_x - C_x)\,dx + (PA_y + QB_y - Cy)\,dy \\ + (PA_z + QB_z - C_z)\,dz + (PA_p + QB_p - C_p)\,dp \\ + (PA_q + QB_q - C_q)\,dq = 0 \\ p\,dx + q\,dy - dz = 0\,.\end{aligned}$$

Eliminating dz and remembering that x, y, p and q are independent there results

$$(1) \qquad PA_x + QB_x - C_x + p(PA_z + QB_z - C_z) = 0$$

$$(2) \qquad PA_y + QB_y - C_y + q(PA_z + QB_z - C_z) = 0$$

$$(3) \qquad PA_p + QB_p = C_p$$

$$(4) \qquad PA_q + QB_q = C_q.$$

Thus the problem reduces to finding sets of solutions of these four partial differential equations. In each of the following paragraphs the solutions obtained are characterized by imposing certain further restrictions on the functions involved.

§1. Let P and Q be each functions of x, y and z only so that the orientation of the transformed element depends solely upon the position of the given element. Then equations (3) and (4) give at once.

$$C = PA + QB + w(x, y, z).$$

Substituting this in (1) and (2) we get

[4] A certain general type of a contact transformation in explicit form was given by Lie in an article Zur Analytischen Theorie der Berührungs Transformationen, Gesammelte Abhandlungen, Dritter Band.

$$(P_x + pP_z)A + (Q_x + pQ_z)B = -(w_x + pw_z)$$
$$(P_y + qP_z)A + (Q_y + qQ_z)B = -(w_y + qw_z).$$

Hence we have

TYPE 1

$$C = PA + QB + w(x, y, z)$$
$$P = P(x, y, z)$$
$$Q = Q(x, y, z).$$

A and B are determined from the last two equations above. (P, Q and w are arbitrary functions of x, y and z except that they are to be chosen so that the two equations are solvable in the usual sense for A and B.)

As an example choose $P = x$, $Q = y$, $w = z$. Then $A = -p$, $B = -q$ and $C = -px - qy + z$. Hence

$$x' = -p$$
$$y' = -q$$
$$z' = -px - qy + z$$
$$p' = x$$
$$q' = y.$$

To obtain a geometrical interpretation eliminate p and q using the first three equations. The result is[5] $xx' + yy' + z - z' = 0$. Denoting by K and K' corresponding geometrical configurations in x, y, z and x', y', z' space respectively, we see that a general K point is transformed into a plane and a general K' point into a plane.

§2. Let P and Q be each functions of p and q so that the orientation of the transformed element depends entirely on the orientation of the given element. Then, in solving equations (1) and (2), p and q may be treated as constants. Letting $u = PA + QB - C$, these equations become $u_x + pu_z = 0$ and $u_y + qu_z = 0$. The common solution is,

[5] An equation $F(x, y, z, x', y', z') = 0$, with certain general restrictions on F, also defines a point to surface contact transformation. See Whittaker's Analytical Dynamics page 294. Likewise two equations $F(x, y, z, x', y', z') = 0$, $G(x, y, z, x', y', z') = 0$ define a point to curve contact transformation. This kind as well as the point to point transformation are not dealt with in this paper.

evidently, $u = w(px + qy - z, p, q)$ so that $C = PA + QB - w$. Substituting this in equations (3) and (4) gives

$$AP_p + BQ_p = w_p, \qquad AP_q + BQ_q = w_q.$$

Hence we have

TYPE 2

$$C = PA + PB - w(px + qy - z, p, q)$$
$$P = P(p, q)$$
$$Q = Q(p, q).$$

A and B are determined from the last two equations above. (P, Q and w are arbitrary functions of their respective arguments except that they are to be chosen so that the two equations are solvable in the usual sense for A and B.)

If we choose $P = p$, $Q = q$ and $w = px + qy - z \pm a\sqrt{1 + p^2 + q^2}$ we get the familiar dilation

$$x' = x \pm \frac{ap}{D}$$
$$y' = y \pm \frac{aq}{D}$$
$$z' = z \mp \frac{a}{D} \qquad D = \sqrt{1 + p^2 + q^2}$$
$$p' = p$$
$$q' = q$$

Eliminating p and q gives $(x - x')^2 + (y - y')^2 + (z - z')^2 = a^2$ so that a general K point is transformed into a sphere and vice versa.

§3. To attack the problem in a more general way, solve (3) and (4) for P and Q and substitute in (1) and (2). We have

$$P = -\frac{\Delta(B, C)}{\Delta(A, B)} \tag{5}$$

$$Q = \frac{\Delta(A, C)}{\Delta(A, B)} \tag{6}$$

$$C_z + pC_x + MC_p + NC_q = 0 \tag{7}$$

$$C_y + qC_x + HC_p + KC_q = 0 \tag{8}$$

where

$$\Delta(B, C) = \begin{vmatrix} B_p, C_p \\ B_q, C_q \end{vmatrix}; \quad \Delta(A, C) = \begin{vmatrix} A_p, C_p \\ A_q, C_q \end{vmatrix}; \quad \Delta(A, B) = \begin{vmatrix} A_p, B_p \\ A_q, B_q \end{vmatrix}.$$

None of these determinants are to vanish identically, a restriction which excludes the point to curve and the point to point transformations. M, N, H and K are defined by the equations.

$$(A_x + pA_z)B_q - (B_x + pB_z)A_q + M\Delta(A, B) = 0$$
$$-(A_x + pA_z)B_p + (B_x + pB_z)A_p + N\Delta(A, B) = 0$$
$$(A_y + qA_z)B_q - (B_y + qB_z)A_q + H\Delta(A, B) = 0$$
$$-(A_y + qA_z)B_p + (B_y + qB_z)A_p + K\Delta(A, B) = 0.$$

These last four equations may be written as follows

$A_x + pA_z + MA_p + RA_q = 0$, R being defined by

$B_x + pB_z + MB_p + RB_q = 0.$

$A_x + pA_z + SA_p + NA_q = 0$, S being defined by

$B_x + pB_z + SB_p + NB_q = 0.$

$A_y + qA_z + HA_p + VA_q = 0$, V being defined by

$B_y + qB_z + HB_p + VB_q = 0.$

$A_y + qA_z + WA_p + KA_q = 0$, W being defined by

$B_y + qB_z + WB_p + KB_q = 0.$

These equations are all of the same form as equations (7) and (8). Therefore, if solutions are obtained for C from (7) and (8), the above equations will yield similar results for A and B provided R, S, V and W are taken equal to N, M, K and H respectively. In what follows only such solutions will be considered. Hence the problem is reduced to finding solutions common to (7) and (8). It should be noted in passing that $C =$ a function of A and B is a solution of (7) and (8); but this is ruled out because it would make J vanish.

If x and y are interchanged and also p and q, the system of equations (7) and (8) remains unchanged in form. This means that for every solution there is a companion solution obtained by making the

above interchanges. But such interchanges constitute merely a change in notation. Hence a solution obtained in this way will not be regarded as a separate type of contact transformation.

The related systems for (7) and (8)

$$dx = \frac{dz}{p} = \frac{dp}{M} = \frac{dq}{N} \tag{9}$$

$$dy = \frac{dz}{q} = \frac{dp}{H} = \frac{dq}{K} \tag{10}$$

will be used frequently in what follows.

§4. We shall now consider the case in which M is a function of x and p only, and N a function of x and q only. We shall assume further that

$$M = -\frac{\theta_x}{\theta_p}, \qquad N = -\frac{\phi_x}{\phi_q}, \qquad \theta_p \neq 0, \qquad \phi_q \neq 0,$$

θ being a function of x and p only and ϕ a function of x and q only. Then two solutions of equation (9) are

$$\theta(x, p) = k, \qquad \phi(x, q) = h,$$

h and k being constants.

To obtain a third solution of (9) solve $\theta(x, p) = k$ for p and let the result be $p = w(x, k)$. Then from equation (9) we have

$$dz - w(x, k)\, dx = 0.$$

The solution is $z - \int w(x, k) = \text{constant}$, the integration to be performed and k replaced by θ. Let the result be denoted by $\mu(x, \theta)$. Later it will be more convenient to start with μ and then determine θ. The relation $\mu_x(x, \theta) = w(x, \theta) = p$ (in which x and θ are treated as independent) shows that this can be done.

We now have $C = F(\theta, \phi, \mu - z)$ as the general solution of (7) and since y is not involved in this equation, we can say that

$$C = F(\theta, \phi, \mu - z, y)$$

is also a solution.

Next we proceed to determine what further restrictions must be imposed on μ and ϕ so that the above solution of (7) will also satisfy equation (8). Substituting and remembering that $\mu_x = p$ we have

$$F_y + (H\mu_\theta\theta_p - q)F_s + H\theta_pF_\theta + K\phi_qF_\phi = 0 \tag{11}$$

in which $s = \mu - z$ and F_y, F_s, F_θ, $F\phi$ denote partial differentiation with respect to the arguments as they explicitly appear. This notation will be used frequently and will not be explained again unless ambiguity arises.

For convenience let

$$H\mu_\theta\theta_p - q = \rho, \qquad K\theta_q = \sigma, \qquad H\theta_p = \tau.$$

At this point the problem is further specialized by considering only those solutions of equation (11) which can be obtained by pairing the members of the related system

$$dy = \frac{ds}{\rho} = \frac{d\phi}{\sigma} = \frac{d\theta}{\tau}$$

and which are subject to the condition that ρ, σ and τ are each functions of s, y, ϕ, θ at most. To determine further the nature of ρ, σ and τ we proceed as follows.

Eliminate H from the equations given above and get $\tau\mu_\theta - \rho = q$. Differentiate this partially with respect to y, z, x, p and q in turn, getting

$$\tau_y\mu_\theta - \rho_y = 0$$

$$\tau_s\mu_\theta - \rho_s = 0$$

$$(\tau\mu_{s\theta} + \tau_s\mu_\theta^2 + \tau_\theta\mu_\theta - \rho_s\mu_\theta - \rho_\theta)\theta_x + (\tau_\phi\mu_\theta - \rho_\phi)\phi_x + \tau\mu_{\theta x} + (\tau_s\mu_\theta - \rho_s)\mu_x = 0$$

$$(\tau\mu_{s\theta} + \tau_s\mu_\theta^2 + \tau_\theta\mu_\theta - \rho_s\mu_\theta - \rho_\theta)\theta_p = 0$$

$$(\tau_\phi\mu_\theta - \rho_\phi)\phi_q = 1.$$

Combining the last four of these equations gives

$$\tau = -\frac{\phi_x}{\phi_q\mu_{\theta x}}.$$

Now suppose that $\mu_{\theta x} = 0$. Then μ_θ = a function of θ only. Hence μ = a function of θ only + a function of x only and μ_x = a function of x only. But μ_x must involve θ, for $\mu_x = p$ when θ is replaced by its equivalent in terms of x and p. Therefore μ_θ must involve x and hence $\mu_{\theta x} \neq 0$. Moreover the above relation shows that τ involves neither y nor s. Hence $\tau_y = 0$ and $\tau_s = 0$ and from the first two of the

above five equations $\rho_y = 0$ and $\rho_z = 0$. Thus τ and ρ are each functions of ϕ and θ at most.

The relation $\tau\mu_\theta - \rho = q$ shows that $\tau\mu_\theta - \rho$ cannot involve p. Hence it cannot involve θ, for θ must involve p; and therefore it must be a function of x and ϕ at most. Further it must involve ϕ, since ϕ is the only one of the three arguments z, ϕ and θ which involves q.

Let $\tau\mu_\theta - \rho = f(x, \phi)$. In determining the form of f, we may regard $\tau\mu_\theta - \rho$ as a function of x, ϕ and θ treated as independent variables. Differentiating partially with respect to x gives $\tau\mu_{\theta x} = fx$. Integrating gives $f = \tau\mu_\theta + \delta(\phi)$. Hence $\rho = -\delta(\phi)$. Then $\tau\mu_\theta = f - \delta = n(x, \phi)$. Differentiate this partially with respect to ϕ and replace μ_θ by its equivalent $\frac{\eta}{\tau}$. Then $\frac{\tau_\theta}{\tau} = \frac{\eta_\phi}{\eta}$. But this says that a function of θ and ϕ is identically equal to a function of x and ϕ. Therefore each member must be a function of ϕ only; that is $\frac{\tau_\phi}{\tau} = \Phi(\phi)$. Integrating gives $\log \tau = \int \Phi \, d\phi +$ a function of θ only.

This may be written in the form $\tau = \dfrac{\alpha(\phi)}{\beta'(\theta)}$ where the prime means differentiation. In similar fashion $\eta = \alpha(\phi)\,\pi(x)$. Then $\mu_\theta = \dfrac{\eta}{\tau} = \beta'(\theta)\,\pi(x)$ and $\mu = \beta(\theta)\,\pi(x) + \lambda(x)$. Note that β cannot be a constant because μ actually involves θ. $\mu_x = \beta\pi' + \lambda'$, whence $\rho = \beta\pi' + \lambda'$ and $\beta = \dfrac{p - \lambda'}{\pi'}$, so that $\mu = \dfrac{(p - \lambda')\pi}{\pi'} + \lambda$. The equation $\tau\mu_\theta - \rho = q$ now becomes $\alpha(\phi)\,\pi(x) - \rho(\phi) = q$. It should be noted at this point that π cannot be a constant because μ_θ must involve x, and that α and ρ must not both be constants because $\alpha\pi - \rho$ must involve ϕ.

To recapitulate, θ, μ and ϕ are now defined as follows:

$$\beta(\theta) = \frac{p - \lambda'(x)}{\pi'(x)}, \qquad \mu = \frac{(p - \lambda'(x))\,\pi(x)}{\pi'(x)} + \lambda(x) \tag{12}$$

$$\alpha(\phi)\,\pi(x) - \rho(\phi) = q.$$

Equation (11) now takes the form

$$F_y + \rho(\theta)\,F_z + \frac{\alpha(\phi)}{\beta'(\theta)}\,F_\theta + \sigma\,F(\phi) = 0. \tag{13}$$

The further treatment of this equation depends on whether or not $\sigma = 0$.

First consider the case in which $\sigma = 0$. Then one solution of the

above equation is F = some function of ϕ. Further solutions may be found by considering the related system

$$dy = \frac{ds}{\rho(\phi)} = \frac{\beta'(\theta)d\theta}{\alpha(\phi)}$$

in which ϕ may be treated as a constant. Solutions are $s - \rho y$ = constant, $s\alpha - \rho\beta$ = constant and $\beta - \alpha y$ = constant.

Thus the general solution of equation (13) is $F = f(\phi, u, v)$ where u and v are any two different selections of the functions $s - \rho y$, $s\alpha - \rho\beta$ and $\beta - \alpha y$. We now have

TYPE 3. A, B and C are each functions of ϕ, u and v; u and v being chosen from the functions

$$\frac{(p - \lambda'(x))\,\pi(x)}{\pi'(x)} + \lambda(x) - z - \rho(\phi)y$$

$$\left(\frac{(p - \lambda'(x))\pi(x)}{\pi'(x)} + \lambda(x) - z\right)\alpha(\phi) - \frac{(p - \lambda'(x))\,\rho(\phi)}{\pi'(x)}$$

$$\frac{(p - \lambda'(x))\,\pi(x)}{\pi'(x)} - \alpha(\phi)y.$$

and ϕ being determined from the relation $\alpha(\phi)\,\pi(x) - \rho(\phi) = q$.

P and Q are determined from equations (5) and (6).

As an example choose

$$\pi = x, \qquad \lambda = 0, \qquad \alpha = 0 \quad \text{and} \quad \rho = -\phi.$$

Then $\phi = q$, and we may choose $u = px + qy - z$, $v = p$. Then $C = f(px + qy - z, p, q)$. A and B are each functions of these same arguments.

This transformation is of particular interest because the pedal transformation, whose equations follow, is readily seen to be a special case.

$$x' = \frac{pu}{D}$$

$$y' = \frac{qu}{D}$$

$$z' = -\frac{u}{D}$$

$$p' = \frac{-xD + 2pu}{(px + qy)D + (2 - D)u}$$

$$q' = \frac{-yD + 2qu}{(px + qy)D + (2 - D)u}$$

where $u = px + qy - z$ and $D = p^2 + q^2 + 1$.

Eliminating p and q using the first three equations we have $x'^2 + y'^2 + z'^2 - xx' - yy' - zz' = 0$. A general K point is transformed into a sphere and a general K' point goes into a plane.

For another example choose $\alpha = \phi$, $\rho = 0$. Then $\phi = \frac{q}{\pi}$ and we may choose $u = \frac{(p - \lambda')\pi}{\pi'} + \lambda - z$, $v = \frac{p - \lambda'}{\pi'} - \frac{qy}{\pi}$. Then A, B and C are functions of ϕ, u and v. Now we can choose A, B and C so that we have the particular transformation

$$x' = \frac{q}{\pi}$$

$$y' = \frac{p - \lambda'}{\pi'} - \frac{qy}{\pi}$$

$$z' = \frac{(p - \lambda')\pi}{\pi'} + \lambda - z$$

$$p' = \pi y$$

$$q' = \pi$$

To obtain a geometrical interpretation, eliminate p and q using the first three equations. The result is $z' + z - \pi x' y - \pi y' - \lambda = 0$. A General K point is transformed into a plane and a general K' point is transformed into a surface whose precise nature depends on π and λ.

Returning to equation (13), let us suppose that $\sigma \neq 0$. Since solutions are to be found from the related system by pairing, σ must be a function of ϕ only. Three solutions are

$$s - \int \frac{\rho d\phi}{\sigma}, \qquad y - \int \frac{d\phi}{\sigma}, \qquad \beta - \int \frac{\alpha d\phi}{\sigma}.$$

Let

$$\int \frac{\rho d\phi}{\sigma} = l, \qquad \int \frac{d\phi}{\sigma} = m, \qquad \int \frac{\alpha d\phi}{\sigma} = n.$$

Then

$$\rho = \frac{l'}{m'}, \qquad \sigma = \frac{1}{m'}, \qquad \alpha = \frac{n'}{m'},$$

Thus we have

TYPE 4. A, B and C are each functions of the expressions

$$\frac{(p-\lambda'(x))\,\pi(x)}{\pi'(x)}+\lambda(x)-z-l(\phi)$$

$$y-m(\phi)$$

$$\frac{p-\lambda'(x)}{\pi'(x)}-n(\phi)$$

ϕ being determined from the relation $n'(\phi)\,\pi(x)-l'(\phi)=qm'(\phi)$.

P and Q are determined from equations (5) and (6).

As an example choose $l=0$, $m=\phi$ $n=\frac{1}{2}\phi^2$. Then $\phi=\dfrac{q}{\pi}$ and

$$C=f\left(\frac{(p-\lambda'(x))\,\pi(x)}{\pi'(x)}+\lambda(x)-z,\ y-\frac{q}{\pi(x)},\ \frac{p-\lambda'(x)}{\pi'(x)}-\frac{q^2}{2[\pi(x)]^2}\right)$$

with similar expressions for A and B. As a special case we have

$$x'=\frac{p-\lambda'(x)}{\pi'(x)}-\frac{q^2}{2[\pi(x)]^2}$$

$$y'=y-\frac{q}{\pi(x)}$$

$$z'=\frac{(p-\lambda'(x))\,\pi(x)}{\pi'(x)}+\lambda(x)-z$$

$$p'=\pi(x)$$

$$q'=-q.$$

Eliminating p and q from the first three equations gives

$$2\pi x'+\pi(y-y')^2-2(z+z')+2\lambda=0.$$

Thus a general K point goes into a parabolic cylinder and a general K' point goes into a surface whose precise nature depends on π and λ.

§5. The case in which M and N in equation (7) are each functions of p and q at most will now be studied. Let $M=\alpha\theta_q$ and $N=-\alpha\theta_p$ where α and θ are each functions of p and q at most and further assume that $\alpha\neq 0$, and $\theta_q\neq 0$.

One solution of equation (13) is $\theta(p, q)=k$. To obtain two more solutions solve this result for q, getting $q=w(p, k)$ and substitute in (13). Then the following solutions may be indicated

$$x - \int \frac{dp}{M} = \text{constant}, \qquad z - \int \frac{pdp}{M} = \text{constant},$$

the integration to be performed and k replaced by θ. Let the results be

$$\int \frac{dp}{M(p, w)} = \mu(p, \theta) \quad \text{and} \quad \int \frac{pdp}{M(p, w)} = \nu(p, \theta).$$

Treating p and θ as independent variables we may write

$$\mu_p = \frac{1}{M}, \qquad \nu_p = \frac{p}{M}, \qquad \nu = \int p\, \mu_p\, dp,$$

the last integration partially with respect to p. Since $\frac{1}{M} \neq 0$ (for then $\Delta(A, B)$ in §3 would vanish), $\mu_p \neq 0$ and $\nu_p \neq 0$. That is μ and ν must each explicitly involve p.

The procedure so far indicates that α and θ are to be chosen and then μ determined. However the relation $\mu_p = \frac{1}{M} = \frac{1}{\alpha\theta_q}$ shows that μ and θ could be chosen and then α determined. This will turn out to be more convenient.

We now have as the solution of equation (7) $C = F(s, t, \theta)$ where $s = x - \mu$ and $t = z - \nu$. Since the variable y does not appear explicitly in (7), $C = F(s, t, \theta, y)$ is also a solution. We now proceed to determine θ so that this solution of (7) will satisfy also equation (8). Substitution gives

$$\begin{aligned} &F_y - [H\mu_p + (H\theta_p + K\theta_q)\mu_\theta]F_s \\ &\quad - [H\nu_p + (H\theta_p + K\theta_q)\nu_\theta - q]F_t + (H\theta_p + K\theta_q)F_\theta = 0. \end{aligned} \tag{14}$$

The related system may be written

$$dy = \frac{ds}{\rho} = \frac{dt}{\sigma} = \frac{d\theta}{\tau}$$

where

$$H\theta_p + K\theta_q = \tau$$

$$\frac{H}{M} + \tau\mu_\theta = -\rho$$

$$\frac{H_p}{M} + \tau\nu_\theta = -\sigma + q$$

remembering that

$$\mu_p = \frac{1}{M} \quad \text{and} \quad \nu_p = \frac{p}{M}.$$

As before, solutions will be sought by pairing the members of the above related system. Then ρ, σ and τ must each be functions of y, s, t and θ at most. Eliminating H between the last three equations gives

$$(\nu_\theta - p\mu_\theta)\tau = p\rho - \sigma + q. \tag{15}$$

Suppose that some one or all of the functions ρ, σ and τ involves s. Solving equation (15) for s gives s equal to a function not containing x. This is contrary to the hypothesis on s. $\therefore$ ρ, σ and τ cannot involve s. A similar argument shows that they cannot involve either t or y. Hence they must be each functions of θ only.

Equation (14) may now be written

$$F_y + \rho F_s + \sigma F_t + \tau F_\theta = 0. \tag{16}$$

The further treatment of this equation depends on whether or not $\tau = 0$.

First assume that $\tau \neq 0$. Then three solutions of the related system are

$$s - \int \frac{\rho d\theta}{\tau}, \qquad y - \int \frac{d\theta}{\tau}, \qquad t - \int \frac{\sigma d\theta}{\tau}.$$

Let

$$\int \frac{\rho d\theta}{\tau} = \alpha, \qquad \int \frac{d\theta}{\tau} = \beta, \qquad \int \frac{\sigma d\theta}{\tau} = \gamma.$$

Then

$$\rho = \frac{\alpha'}{\beta'}, \qquad \tau = \frac{1}{\beta'}, \qquad \sigma = \frac{\gamma'}{\beta'}.$$

Equation (15) now takes the form

$$\nu_\theta - p\mu_\theta = \rho\alpha' - \gamma' + q\beta'. \tag{17}$$

Hence we can write down

TYPE 5. A, B and C are each functions of the expressions

$$x - \mu(\rho, \theta) - \alpha(\theta)$$
$$y - \beta(\theta)$$
$$z - \int p\, u_\rho\, d\rho - \gamma(\theta).$$

θ being determined from the relation (17).

P and Q are determined from equation (5) and (6). As an example let

$$\mu = \frac{\mp ap\theta}{\sqrt{1+p^2}\sqrt{1+\theta^2}}, \qquad \alpha = 0, \qquad \beta = \frac{\mp aq}{\sqrt{1+\theta^2}}, \qquad \gamma = 0.$$

Then

$$\nu = \int \frac{\mp ap^2\theta d\rho}{\sqrt{1+p^2}\sqrt{1+\theta^2}} = \frac{\pm a\theta}{\sqrt{1+p^2}\sqrt{1+\theta^2}}$$

and θ is determined from equation (17) as follows

$$\pm\left(\frac{1}{\sqrt{1+p^2}} + \frac{p^2}{\sqrt{1+p^2}}\right)\frac{a}{(1+\theta^2)^{3/2}} \cdot \frac{(1+\theta^2)^{3/2}}{a\theta} = q$$

$$\theta = \frac{\pm\sqrt{1+p^2}}{q}.$$

Thus

$$\mu = \frac{\mp ap}{\sqrt{1+p^2+q^2}}, \qquad \nu = \frac{\pm a}{\sqrt{1+p^2+q^2}}, \qquad \beta = \frac{\mp aq}{\sqrt{1+p^2+q^2}}.$$

Therefore A, B and C are each functions of the expressions

$$x \pm \frac{ap}{D}, \qquad y \pm \frac{aq}{D}, \qquad z \mp \frac{a}{D}$$

where

$$D = \sqrt{1+p^2+q^2}.$$

Again the dilation mentioned in §2 is readily seen to be a special case of this transformation.

Returning to equation (16) suppose that $\tau = 0$. One solution of this equation is $F =$ some function of θ. The other solutions may be obtained from the related system

$$dy = \frac{dz}{\rho} = \frac{d\theta}{\sigma}$$

in which ρ and σ are treated as constants. Solutions are $\rho y - s =$ constant, $\sigma y - t =$ constant, $\sigma s - \rho t =$ constant. Equation (15) now has the form

$$p\rho(\theta) - \sigma(\theta) + q = 0. \tag{18}$$

Therefore we have

TYPE 6. A, B and C are each functions of θ, u and v; u and v being chosen from the expressions

$$\rho(\theta)y - x + \mu(p, \theta),$$

$$\sigma(\theta)y - z + \int p\mu_p\, dp,$$

$$\sigma(\theta)(x - \mu(p, \theta)) - \rho(\theta)(z - \int p\mu_p\, dp)$$

and θ being determined from the relation (18).

P and Q are found from equations (5) and (6).

As an example let $\rho = 0$, $\sigma = \theta$, $\mu = p + \theta$. Then $\nu = \frac{1}{2}p^2$, $\theta = q$. If we choose $u = x - p - q$ and $v = z - \frac{1}{2}p^2 - qy$ we have

$$C = f(x - p - q, z - \tfrac{1}{2}p^2 - qy, q),$$

with similar expressions for A and B. As a special case we can write

$$x' = x - p - q$$

$$y' = q$$

$$z' = z - \tfrac{1}{2}p^2 - qy$$

$$p' = p$$

$$q' = p - y.$$

Eliminating p and q from the first three equations gives

$$2(z' - z) + (x - x' - y')^2 + 2y'y = 0.$$

A general K point is transformed into a certain quadric surface. Likewise a general K' point goes into a quadric surface.

ANALYTIC AFFINE TRANSFORMATIONS IN EUCLIDEAN SPACE OF $2n$ DIMENSIONS

By A. E. Staniland[1]

In a previous paper[2] the author discussed analytic rotations in a flat four space, representative of the plane of two complex variables. A generalization of the results of that paper is embodied in the present one.

Consider a complex linear n-space, Σ_n, that is, a linear manifold of points to each of which is assigned a set of n complex coördinates, $z_r \equiv x_{2r-1} + ix_{2r}$, $r = 1, 2 \cdots n$. We take the x's as real rectangular coördinates of a point in Euclidean space of $2n$ dimensions, S_{2n}. By an analytic function, $w = F(z_1, z_2 \cdots z_s)$ of an arbitrary number of complex arguments we shall understand, according to the definition which has been in use since Cauchy, a function which is analytic in each argument separately. An important consequence of this is that w is a continuous function of its arguments taken collectively. The real representation in S_{2n} of the complex configuration in Σ_n corresponding to a set of $k < n$ independent analytic functions, the total number of dependent and independent variables not exceeding n, is a continuous analytic surface, usually a hypersurface. Finally an analytic transformation of the real space will be defined as one that carries an analytic surface into an analytic surface. It is our purpose to determine necessary and sufficient conditions that an affine transformation of S_{2n} shall be analytic.

It will be sufficient to consider a single analytic function,

$$w = \varphi(z_1, z_2, \cdots z_{n-1}),$$

of the maximum number of independent variables possible for the space, Σ_n. We may, without loss of generality, take w in its implicit form:

$$F(z_1, z_2 \cdots z_n) = 0, \tag{1}$$

where $w \equiv z_n$ and $\partial F/\partial z_n \neq 0$ in the domain of definition of w. Subject the $2n$ real x_i of (1) to the transformation

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[2] A Note on Analytic Rotations in Euclidean Space of Four Dimensions. *Journal of Mathematics and Physics*, vol. xv, no. 1, Feb. 1936.

(2) $\qquad x_i = \Sigma a_{ij}x'_j + b_i, \qquad a_{ij}, b_i$ real and $|a_{ij}| \neq 0.$

It is clear that the translation

$$x_i = x'_i + b_i$$

leaves an analytic surface analytic. Hence it is only necessary to consider the transformation

(3) $\qquad x_i = \Sigma a_{ij}x'_j, \qquad |a_{ij}| \neq 0.$

Under (3), $w = F(z_1, z_2 \cdots z_n) = 0$ becomes

(4) $\qquad F(\Sigma a_{1j}x'_j + i\Sigma a_{2j}x'_j, \cdots, \Sigma a_{2n-1,j}x'_j + i\Sigma a_{2n,j}x'_j) = 0.$

If (4) is to define the new $w' = x'_{2n-1} + ix'_{2n}$ as an analytic function of new $z'_k = x'_{2k-1} + ix'_{2k}$, it is necessary that the generalized Cauchy-Riemann equations continue to hold. These are

(5) $$\frac{\partial x'_{2n-1}}{\partial x'_{2k-1}} = \frac{\partial x'_{2n}}{\partial x'_{2k}} \qquad \frac{\partial x'_{2n-1}}{\partial x'_{2k}} = -\frac{\partial x'_{2n}}{\partial x'_{2k-1}}$$

$$(k = 1, 2, \cdots n-1).$$

For convenience set $a_{ij} \equiv (i, j)$. We have then, formally, from (4)

(6) $$\frac{\partial x'_h}{\partial x'_m} = -\frac{\Sigma\lambda_r\{(2r-1, m) + i(2r, m)\}}{\Sigma\lambda_r\{(2r-1, h) + i(2r, h)\}}$$

where $\lambda_r = \partial F/\partial z_r$

Taking, in (6), successively,

$$h = 2n-1, \qquad m = 2k-1,$$
$$h = 2n, \qquad m = 2k,$$
$$h = 2n-1, \qquad m = 2k,$$
$$h = 2n, \qquad m = 2k-1,$$

and forming equations (5) we get, after clearing of fractions, a set of $(n-1)$ pairs of complex homogeneous quadratic equations in the $\lambda_r (\lambda_r \neq 0)$. The coefficients of the λ_r^2 are of two forms, corresponding to the first and second of equations (5), namely

(7) $$AD - BC,$$
$$BD + AC$$

where

$$A \equiv (2r-1, 2k-1) + i(2r, 2k-1)$$
$$B \equiv (2r-1, 2k) + i(2r, 2k)$$
$$C \equiv (2r-1, 2n-1) + i(2r, 2n-1)$$
$$D \equiv (2r-1, 2n) + i(2r, 2n)$$
$$(r = 1, 2 \cdots n; k = 1, 2, \cdots n-1).$$

The coefficients of $\lambda_r\lambda_s$ $(r \neq s)$ have the forms

$$\begin{aligned} &AD' + A'D - BC' - B'C \\ &BD' + B'D + AC' + A'C \end{aligned} \tag{8}$$

where A', B', C', D' are the same functions respectively of the index s as A, B, C, D are of the index r. The λ's are arbitrary since w was assumed to be any analytic function of its arguments. Hence the coefficients (7) and (8) vanish and we obtain

$$\begin{aligned} AD - BC &= 0 \\ BD + AC &= 0 \\ AD' + A'D - BC' - B'C &= 0 \\ BD' + B'D + AC' + A'C &= 0. \end{aligned} \tag{9}$$

In (9), let A_1, B_1, C_1, D_1, be values of A, B, C, D corresponding to any particular value r_1 of r on the range from 1 to n. Then if

$$A_1^2 + B_1^2 \neq 0,$$

we have $C_1 = D_1 = 0$; whence also

$$A_1D' - B_1C' = 0,$$
$$B_1D' + A_1C' = 0,$$

involving $C' = D' = 0$. But $C_1 = D_1 = C' = D' = 0$ entails the vanishing of the last two columns of $[a_{ij}]$ contrary to the hypothesis, $|a_{ij}| \neq 0$. Hence $A_1^2 + B_1^2 = 0$. Since r_1 was any value of r on the range from 1 to n, we have

$$A^2 + B^2 = 0 \qquad (r = 1, 2 \cdots n). \tag{10}$$

Similarly, the assumption $C_1^2 + D_1^2 \neq 0$ leads to $A_1 = B_1 = A' =$

$B' = 0$ which implies, for each value of k, the vanishing of the $(2k-1)$th and $2k$-th columns of $[\,a_{ij}\,]$. Hence

$$C^2 + D^2 = 0 \qquad (r = 1, 2 \cdots n). \tag{11}$$

Equations (10) and (11) are equivalent to

$$(2r, 2k) = \pm(2r-1, 2k-1)$$
$$(2r, 2k-1) = \pm(2r-1, 2k)\,.$$

Equations (9), which must be satisfied identically, may be used to determine the correct association of signs. We thus obtain the two forms of necessary and sufficient restrictions,

I:
$$(2r, 2k) = (2r-1, 2k-1)$$
$$(2r, 2k-1) = -(2r-1, 2k)$$

II:
$$(2r, 2k) = -(2r-1, 2k-1)$$
$$(2r, 2k-1) = (2r-1, 2k)\,.$$

In Σ_n, (1) assumes, under I, the complex affine form

$$\begin{aligned} z_i &= \Sigma A_{ij} z_j' + B_i \\ z_j' &= x_{2j-1}' + i x_{2j}' \\ A_{kj} &= (2k-1, 2j-1) - i(2k-1, 2j) \\ B_i &= b_{2i-1} + i b_{2i}\,. \end{aligned} \tag{12}$$

Under II each z_i is replaced by $\bar{z}_i$.

Certain specializations of the A_{ij} will occur to the reader. In the sequel, however, we shall be concerned only with the unitary transformations of types I and II which are respectively,

$$U\colon z_i = \Sigma A_{ij} z_j' \quad \text{and} \quad \bar{U}\colon \bar{z}_i = \Sigma A_{ij} z_j'$$

for both of which

$$\Sigma A_{ij} \bar{A}_{kj} = \epsilon_{ik};\ \epsilon_{ik} = 1 \text{ when } i = k;$$
$$\epsilon_{ik} = 0 \text{ when } i \neq k;\ |\,A_{ij}\,| = e^{i\varphi},\ \varphi \text{ real.}$$

Transformation U. This is the complex form of a rotation of S_{2n} about the origin. It depends upon $2n^2 - n - 2\binom{n}{2} = n^2$ independent real parameters. These parameters may be separated into two inde-

pendent sets of $n^2 - n$ and n parameters, with distinct geometric interpretations.

The maximum number of mutually completely perpendicular analytic planes through the origin of S_{2n} that it is possible for one to consider is n. Such a set of planes will be called briefly a set of rotation planes. The complex equations of a set of rotation planes are the n different sets of $n - 1$ equations that it is possible to select from the n equations

$$(13) \quad \Sigma M_{ij} z_j = 0; \quad \Sigma M_{ij} \bar{M}_{kj} = 0 \text{ for } i \neq k; \quad [M_{ij}] \text{ of rank } n.$$

In each of these equations there are $n - 1$ ratios of the M_{ij}. The equations contain, therefore, $n(n-1)$ complex constants restricted by $\binom{n}{2}$ complex conditions. Hence a set of rotation planes depends upon $2\left[n^2 - n - \binom{n}{2}\right] = n^2 - n$ independent real parameters.

Let T, of matrix $[P_{ij}]$, be a transformation of type U carrying $z_k = 0$ into $\Sigma M_{kj} z' = 0$. Hence $P_{kj} = \rho_k M_{kj}$, each ρ_k a complex constant. Now $z_k = 0$ is the complex form of the unique linear $(2n-2)$-space in S_{2n} which is completely perpendicular to the plane of x_{2k-1} and x_{2k}. Likewise $\Sigma M_{kj} z'_j = 0$ is the complex form of the unique linear $(2n-2)$-space which is completely perpendicular to the plane, π_k, into which the plane of x_{2k-1} and x_{2k} is carried by T. Hence T carries the set of coördinate rotation planes, $z_k = 0$ $(k = 1, 2, \cdots n)$, the equations to be taken in sets of $n - 1$, into the set of rotation planes $\{\pi_k\}$ defined similarly by (13), each plane π_k corresponding to the plane of x_{2k-1} and x_{2k} in the transformation.

A rotation in complex form of the plane of x_{2k-1} and x_{2k} is

$$R_k: \qquad z_k = e^{i\varphi_k} z'_k; \qquad z_h = z'_h, \qquad h \neq k.$$

A rotation of the set of coördinate analytic planes upon themselves is expressed by the product $U_0 = \Pi R_k$ in which the R_k are mutually commutative. If, now, we allow the M_{ij} to assume all values consistent with (13), we have for the complex form of the general analytic rotation of type I, $T^{-1} U_0 T$. For the matrix of T^{-1} we have $a^{-1}[p_{ji}]$, where $a = |P_{ij}|$ and p_{ij} = cofactor of P_{ij} in $|P_{ij}|$. Now $p_{ij} = a\bar{P}_{ij}$, a property of every unitary matrix. Hence the matrix of T^{-1} assumes the form $[\bar{P}_{ji}]$. Finally for $T^{-1} U_0 T$ we obtain the matrix

$$(14) \qquad [A_{ij}] = [\Sigma \rho_k \bar{\rho}_k \bar{M}_{ki} M_{kj} e^{i\varphi_k}]$$

each ρ_k satisfying $\rho_k \bar{\rho}_k (\Sigma M_{kj} \bar{M}_{kj}) = 1$.

Thus the A_{ij} of U are exhibited explicitly as functions of the mutually independent sets $\{M_{ij}\}$ and $\{\varphi_k\}$ of dimensionalities $n^2 - n$ and n respectively. The group of transformations, U, is seen to consist of $\infty^{n(n-1)}$ simply isomorphic subgroups of ∞^n members each, the canonical form of the transformation being

$$U_0 = \Pi R_k.$$

The determinant of the right member of (14) is $e^{i\theta}$, $\theta = \Sigma\varphi_k$. Hence φ = amplitude $|A_{ij}| = \Sigma\varphi_k$. The matrix of the real transformation in S_{2n} corresponding to U_0 consists of n blocks of the form

$$\begin{vmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{vmatrix}$$

along the principal diagonal. The matrix of the general rotation of type I is the transform of this by the real matrix which corresponds to T. Hence, with restrictions I,

$$|a_{ij}| = \Pi(\cos^2\varphi_k + \sin^2\varphi_k) = \cos^2\Sigma\varphi_k + \sin^2\Sigma\varphi_k = 1.$$

The Transformation $\bar{U}$. This is a rotation in S_{2n} of type I, accompanied by n reflections, one in each of the rotation planes. Hence $\bar{U}$ is a rotation if and only if n is even. When n is odd, $|a_{ij}| = -1$. Transformations of type $\bar{U}$ do not form a group since the product of two such is of type U.

A real multiplicative form for $\bar{U}$, in which the multipliers have the values ± 1, can be easily obtained. Consider the transformation, in complex form, of the plane π_k in S_{2n}:

$$R_k': \qquad z_k = e^{i\phi_k} z_k', \qquad z_h = z_h', \qquad h \neq k.$$

Now R_k' leaves absolutely invariant the line L_k: $x_{2k} = -\tan\frac{\phi_k}{2} \cdot x_{2k-1}$. The only other invariant line in π_k (not point for point) is $\bar{L}_k$: $x_{2k} = \cot\frac{\phi_k}{2} \cdot x_{2k-1}$, which is perpendicular to L_k. Hence L_k and $\bar{L}_k$ may be taken for new x_{2k-1} and x_{2k} axes. A rotation of type U_0 carrying the old x_{2k-1} and x_{2k} axes into L_k and $\bar{L}_k$ respectively is

$$T_k: \qquad z_k = e^{-i\frac{\phi_k}{2}} \cdot z_k', \qquad z_h = z_h', \qquad h \neq k.$$

Then we have

$$T_k^{-1}R_k'T_k = V_k: \qquad z_k = z_k', \qquad z_h = z_h', \qquad h \neq k.$$

The desired multiplication in complex form is

$$U_0' = \Pi V_k \qquad (k = 1, 2, \cdots n),$$

whence, for the real coördinates, we obtain

$$\begin{aligned} x_{2k-1} &= x_{2k-1}' \\ x_{2k} &= -x_{2k}' \end{aligned} \tag{15}$$

$$(k = 1, 2, \cdots n).$$

Corresponding to various combinations of the axes, pair-wise, to determine sets of n mutually completely perpendicular planes, we may form from (15) equation pairs of the types

$$\begin{aligned} x_{2l-1} &= x_{2l-1}' \\ x_{2m-1} &= x_{2m-1}' \end{aligned} \qquad l \neq m$$

$$\begin{aligned} x_{2l} &= -x_{2l}' \\ x_{2p} &= -x_{2p}' \end{aligned} \qquad l \neq p$$

each of determinant $+1$. Such pairs will completely exhaust the equations (15) when n is even. When n is odd there will be an additional pair of the type

$$\begin{aligned} x_{2r-1} &= x_{2r-1}' \\ x_{2r} &= -x_{2r}' \end{aligned}$$

of determinant -1.

There is thus no distinction of type, *in a strictly geometric sense*, between $\bar{U}$ (n even) and U, or between $\bar{U}$ (n odd) and the indirect orthogonal transformations of the classical theory.

Summary of this paper. We have employed a flat space of $2n$ dimensions, S_{2n}, as a real representation of a complex linear n-space, Σ_n, by establishing a correspondence between the complex coördinates of the latter space and ordered pairs of real coördinates of the former. Sets of analytic functions defined in Σ_n are represented by continuous surfaces, called analytic, in S_{2n}. An analytic transformation of the real space is one that leaves invariant the aggregate of its analytic surfaces. By insisting that the extended Cauchy-Riemann equations continue to hold for all functions initially analytic but otherwise op-

tional we have obtained necessary and sufficient conditions that an affine transformation of S_{2n} be analytic. The analytic affine transformations of S_{2n} correspond exactly to two types of affine transformation of Σ_n:

$$A\colon z_k = \Sigma A_{kj} z'_j + B_k; \qquad \bar{A}\colon \bar{z}_k = \Sigma A_{kj} z'_j + B_k. \qquad |A_{kj}| \neq 0.$$

Included in A and $\bar{A}$ are the unitary types U and $\bar{U}$ which are reducible, by a change of coördinate planes, to the multiplicative forms

$$U_0\colon z_k = e^{i\phi_k} z'_k \quad \text{and} \quad \bar{U}_0\colon \bar{z}_k = e^{i\phi_k} z'_k \quad (k = 1, 2 \cdots n).$$

The transformations U and $\bar{U}$ leave invariant a set of n mutually completely perpendicular planes $\{\pi_k\}$ in S_{2n}. Under U each plane π_k is turned through the angle ϕ_k; under $\bar{U}$ each plane rotation through ϕ_k is accompanied by a single reflection. In each case the amplitude of $|A_{kj}| = \Sigma\phi_k$. A still simpler form for $\bar{U}$ is

$$\bar{U}'_0\colon \bar{z}_k = z'_k \qquad (k = 1, 2, \cdots n),$$

obtained by referring the transformation to new axes consisting of n invariant pairs of perpendicular lines, one pair in each of the invariant planes.